Print your name: ____________________

Midterm Examination — Spring 2020
Econ 705 — Page Total = 5

1. Comparative Definitions: 10 points each

DO **ALL** QUESTIONS. QUESTIONS ARE EQUALLY WEIGHTED. For each term, provide

- a mathematical statement,
- an intutively helpful verbal restatement, and
- an illustrative *example*.

(Be sure to explain why your example is in fact an example.) Then compare and contrast the two terms in each problem, highlighting the points of similarity and difference.

DEF1. topology vs algebra of sets

DEF2. acyclicity (AAC) vs transitivity

DEF3. substitutive set building vs selective set building

DEF4. undominated (maximal) set vs dominant (best) set

2. Problems: 20 points each

DO **ALL** QUESTIONS. QUESTIONS ARE EQUALLY WEIGHTED.

SA1. (a) Use truth tables to define logical negation, logical conjunction, logical disjunction, and material implication. (You may use a combined table.)

(b) Use a truth table to prove that a material implication is logically equivalent to a particular logical disjunction.

(c) Use a truth table to prove that a material implication is logically equivalent to its contrapositive.

(d) Describe the role of logical equivalence in logical proof.

SA2. Using elements from $X = \{x, y, z\}$, we have the collection of ordered pairs $R = \{(x, y), (y, z), (z, x)\}$.

(a) R defines a binary relation in X. Explain why. (Carefully define all terms.)

(b) Is this binary relation reflexive, symmetric, or transitive? (Define each term, and justify your answer by referring explicitly to your definition.)

(c) Draw a graph plot representing this binary relation.

(d) Construct a boolean matrix representation of this binary relation. Carefully explain how your matrix relates to the relation.

(e) Carefully explain how to use boolean matrix operations to produce a representation of the reflexive closure, symmetric closure, and transitive closure of *any* binary relation. (Be sure to define all terms.)

(f) Use matrix operations to produce a representation of the reflexive closure, symmetric closure, and transitive closure of R. (Show any relevant matrix work. Doing things intelligently is valued; e.g., show the required matrix algebra, but don't do clearly unnecessary matrix operations.) Provide a simple graph plot of each result.

SA3. Consider a transitive binary relation R on a set X. Show that the symmetric component I must also be transitive. Show that the asymmetric component P must also be transitive. Also show that $(x\ P\ y \wedge y\ I\ z) \implies x\ P\ z$ and that $(x\ I\ y \wedge y\ P\ z) \implies x\ P\ z$. Make sure you provide the reason for each step in the your proofs.

SA4. Consider the binary relation B on a set X.

- Provide necessary and sufficient conditions for the existence of a B-maximal element on every non-empty *finite* subset of X. Prove that your conditions are necessary and sufficient.
- Prove the compactness of the maximal set in any finite set.
- Prove Walker's theorem. Make sure your math is rigorous and that you define *all* terms. Be sure to give a *detailed* and intuitive explanation of each step in your proof.

3. Multiple Choice: 2 points each

DO ALL OF THE FOLLOWING QUESTIONS. ALL QUESTIONS ARE EQUALLY WEIGHTED. (Please write the question number and answer letter *in your bluebook*.)

MC1. For arbitrary sets P and Q, we know

(a) $(P \cap Q)^c = (P^c \cup Q^c)$
(b) $(P \cap Q)^c = (P^c \cap Q^c)$
(c) $(P \cap Q)^c = (P \cup Q)^c$
(d) a. and b.
(e) all of the above

MC2. Using a finite set of n elements, how many different sequences of length k can we produce? (Here $k \leq n$.)

(a) $\frac{n!}{k!(n-k)!}$
(b) $n!/(n-k)!$
(c) $n!/k!$
(d) $n!$
(e) none of the above

MC3. Using a finite set of n elements, how many distinct sets of cardinality k can we produce? (Here $k \leq n$.)

(a) $\frac{n!}{k!(n-k)!}$
(b) $n!/(n-k)!$
(c) $n!/k!$
(d) $n!$
(e) none of the above

MC4. Consider the topological space (X, τ) where $X = \{1, 2, 3, 4\}$. You are given that $\{1\}$ and $\{4\}$ are open sets in X. Which must be true:

(a) $\{1, 4\}$ is open
(b) $\{2, 3\}$ is closed
(c) $\{1\}$ is closed
(d) a. and b.
(e) all of the above

MC5. As a thought experiment, imagine flipping a coin n times, keeping track of the result from each flip. For $k = 0, \ldots, n$, define events $E_k = \{\omega \in \Omega | \, \text{total}(\omega) = k\}$, where total($\omega$) is the sum over the zeros and ones that are the elements of the resulting n-tuple. Consider the set $\mathcal{A}$ comprising all possible ways to form unions of these events, plus the empty set.

(a) $\mathcal{A}$ is closed under pairwise union.

(b) $\mathcal{A}$ is closed under complementation.

(c) $\mathcal{A}$ is a partition of the sample space.

(d) a. and b.

(e) all of the above

MC6. The Kolmogorov probability axioms stipulate that a probability measure is

(a) positive semi-definite

(b) normalized

(c) sigma-additive

(d) a. and b.

(e) all of the above

MC7. The Kolmogorov probability axioms imply that a probability measure $\mathbf{P}$ will have which of the following properties?

(a) $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

(b) monotonicity: if $E_1 \subseteq E_2$, then $\mathbf{P}(E_1) \leq \mathbf{P}(E_2)$.

(c) inclusion-exclusion: $\mathbf{P}(E_1 \cup E_2) = \mathbf{P}(E_1) + \mathbf{P}(E_2) - \mathbf{P}(E_1 \cap E_2)$

(d) b. and c.

(e) all of the above

MC8. R is a transitive binary relation in X. It must be true that

(a) $R^2 \subseteq R$

(b) $R^3 \subseteq R$

(c) $\cup_{i=1}^{\infty} R^i \subseteq R$

(d) a. and b.

(e) all of the above

MC9. R is a transitive binary relation in X with symmetric component I. Then it must be true that

(a) $xRy \wedge yIz \implies xIz$

(b) $xIy \wedge yIz \implies xIz$

(c) $xIy \wedge yRz \implies xIz$

(d) a. and b.

(e) all of the above

MC10. R is a transitive binary relation in X with asymmetric component P. Then it must be true that

(a) $xRy \wedge yRz \implies xRz$
(b) $xRy \wedge yPz \implies xPz$
(c) $xPy \wedge yRz \implies xPz$
(d) a. and b.
(e) all of the above

MC11. R is a transitive and symmetric binary relation on X. Then it must be true that

(a) $(x, y) \in R \implies (x, x) \in R$
(b) $(y, x) \in R \implies (x, x) \in R$
(c) R is reflexive.
(d) a. and b.
(e) all of the above

MC12. R is a binary relation in the finite set X, and there are no cycles in its asymmetric component. Then it must be true that

(a) there exists at least one R-best element in X
(b) there exists at least one R-maximal element in X
(c) every R-maximal element is an R-best element
(d) a. and c.
(e) all of the above

MC13. R is a transitive binary relation in X, x is an R-best element, and yRx. Then it must be true that

(a) y is a best element
(b) y is a maximal element
(c) xRy
(d) a. and b.
(e) all of the above

MC14. R is a transitive binary relation in X, x is an R-maximal element, and yRx. Then it must be true that

(a) y is a best element
(b) y is a maximal element
(c) xRy
(d) b. and c.
(e) all of the above

MC15. Let R be a *complete* binary relation in X. Let $M(X, R)$ be the set of R-maximal elements in X, and let $C(X, R)$ be the set of R-best elements in X. Which of the following are are always true?

(a) $M(X, R) \subseteq C(X, R)$
(b) $C(X, R) \subseteq M(X, R)$
(c) $M(X, R) = C(X, R)$
(d) all of the above
(e) none of the above

MC16. Consider the following "Putative Theorem:" Suppose R is a relation in the set A. If R is symmetric and transitive, then R is reflexive.
"Putative Proof:" Let x be an arbitrary element of A. Let y be any element of A such that $x \mathrel{R} y$. Since R is symmetric, it follows that $y \mathrel{R} x$. But then by transitivity, since $x \mathrel{R} y$ and $y \mathrel{R} x$, it follows that $x \mathrel{R} x$. Since x was arbitrary, it follows that $x \mathrel{R} x$ forall x in A. That is, R is reflexive.
This proof is

(a) correct if the domain of defintion is A
(b) correct if the range is A
(c) correct if A is empty
(d) a. and b.
(e) all of the above

MC17. Suppose we consider $R = \varnothing$ to be a binary relation in the *non-empty* set X. Which of the following are true?

(a) R is transitive.
(b) R is symmetric.
(c) R is reflexive.
(d) a. and b.
(e) all of the above

MC18. Let R_t be the transitive closure of the binary relation R. Then we know

(a) if R is reflexive then so is R_t
(b) if R is symmetric then so is R_t
(c) if R is transitive then $R = R_t$
(d) a. and b.
(e) all of the above

MC19. R is a preorder on X. Then it must be true that

(a) $x \in \mathrm{Dom}(R) \implies (x, x) \in R$
(b) $x \in \mathrm{Ran}(R) \implies (x, x) \in R$
(c) R is reflexive.
(d) a. and b.
(e) all of the above

MC20. We have 100 purchase observations for bundles of 5 goods. Each observation is a price vector and a quantity vector. We create two matrices: each row of X has quantities purchased and the corresponding row of P is the prices at which they were purchased. Which of the following **WL** expressions gives the total spending for each observation?

(a) `P.X`
(b) `P*X`
(c) `Total/@(P.X)`
(d) `Total/@(P*X)`
(e) c. and d.