Mathematical Economics: Homework

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Collected Assignments Assignments will be listed below as they are assigned. Each assignment is due before the start of the subsequent class, unless otherwise announced. Each assignment should include a header that lists the homework number, your name, and an acknowledgment of your group members (or anyone else who provided useful help).

Homework submissions must be typed in a Mathematica notebook and submitted in PDF format. (PDF Creation Hint: disable "Dynamic Updating" (`Menu/Evaluation/Dynamic Updating Enabled`) before printing to PDF.) Make sure you do a Mathematica tutorial before attempting to write up your homework! A single file should contain your answers to all the exercises for a homework. Computational problems should include helpful comments on your Mathematica code.

Homework is individual work, not group work. You may verbally discuss problems with other students, but for the graded problems you **must not** look at their work, and you **must not** show your work to them. **Looking at or sharing work on graded problems is an Academic Integrity Violation that can lead to program dismissal.** In contrast, full collaboration on the optional (ungraded) problems is unrestricted and is encouraged.

Hints in Discussion Sections: Be sure to read the discussion sections that I provide for the computational problems. In addition to providing hints, they sometimes include details about the problem *requirements*.

Computational Exercises: Some *computational* exercises below could easily be done with a calculator or even by hand. In such cases you need to provide the code, however trivial.

WL Programming: The online **WL** documentation is excellent. E.g., `http://www.wolfram.com/broadcast/video.php?channel=89&video=409`.

Timely Submission: Assignments are due *before* the next class starts. Turn in a PDF file created from Mathematica. Use the sectioning facilities of Mathematica notebooks: use a separate subsection for each problem. Please pay attention to both the general and the language-specific discussion that I append to many problems. Your filename should combine the homework number and *your* last name: e.g., `hw01-Lastname.pdf`. (Never use spaces or any additional punctuation in your file names.) Submit this PDF via Blackboard.

Reminder: A crash upon PDF creation is rare but not unknown. So *first* save your work. Then, after saving your work, try saving it as PDF. You may have better success saving as PDF if you first go to the Evaluation menu and disable Dynamic Updating.

Reminder: The syllabus readings are required. Do the readings *before* attempting the homework. Additionally, be sure to read the hints for each problem.

You must type your homework in Mathematica, but use of Mathematica commands is typically *not* needed for the analytical exercises. (As opposed to the computational exercises.) Be sure that you are working with my Introductory Mathematica Resources. (If you skip the tutorials, you will *of course* work very slowly.) Use the Canvas Discussions to raise questions about the homework or about Mathematica.

Reminder: Course policy requires you to use an external back up or your homework files, to guard against computer failure or theft. Common choices are Google Drive for Desktop, OneDrive, or iCloud Drive.

Reminder: Use the sectioning facilities of Mathematica notebooks: use a separate section (Alt+4) or subsection (Alt+5) for each problem. Be sure to use text cells (not input cells) for your verbal explanations and mathematical proofs. Be sure that you have mastered entering text and mathematics in text cells (e.g., via the Hands on Start to Mathematica.)

Assignments

Academic Integrity Statement: As described in class, I encourage discussion of the homework. However, submitted homework must be written up on your own, without looking at solutions produced by others or by an AI. So near the top of your submitted homework, please include the following signed statement:

> "I completed this assignment without looking at the work on these problems of other students, and I did not show my work on these problems to other students."

Reminder: You must turn on Mathematica's automatic backup and also back up to the cloud.

Homework 13

This “homework” is for review purposes only. It is *not* collected or graded.

Exercise 13.1 Hoy p.481 section 11.5 #1,5
Hoy 12.1 #1,3
Hoy 12.2 #1,3,5

Exercise 13.2 For each of the following real functions,

- produce the gradient (i.e,, the vector of first-order partial derivatives)
- find the stationary points (if any)
- produce the hessian matrix (i.e., the matrix of second-order partial derivatives) and check for definiteness.
- Wherever you find a stationary point, use the hessian to classify it (evaluating the the hessian definiteness at the stationary points).
- draw a helpful 3-D plot and relate it to your results

a. $\langle x, y\rangle \mapsto x + y$

b. $\langle x, y\rangle \mapsto x^2 + y^2$

c. $\langle x, y\rangle \mapsto (x + y)^2$

d. $\langle x, y\rangle \mapsto xy$

e. $\langle x, y\rangle \mapsto \sqrt{xy} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

f. $\langle x, y\rangle \mapsto \sqrt{xy} - 2x - 3y : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

g. $\langle x, y\rangle \mapsto \sqrt[3]{xy}$

h. $\langle x, y\rangle \mapsto \sqrt[3]{xy} - (x + y)/2$

General Discussion: Make sure you can find the gradient and hessian by hand. Use computer algebra to check your work.

WL Discussion: The `D` command is very powerful. For example, given a binary real function f, `D[f[x,y],x,y]` produces the gradient and and `D[f[x,y],x,y,2]` produces the hessian. (This is explained in the *Details* section of the command’s documentation.) After you define a function f you can check your work with

```
grad = Grad[f[x, y], {x, y}]
Solve[0 == grad, {x, y}]
Plot3D[f[x, y], {x, -1, 1}, {y, -1, 1}]
```

Note that if there is no solution, `Solve` will return an empty list. May sure you can explain why this happens when/if it does.

Exercise 13.3 (Linear Comparative Statics Review) Consider the two structural equations:

$$
\begin{aligned}
y &= a - b(i - \pi) \\
m &= ky - hi
\end{aligned}
\tag{13.1}
$$

Here y is aggregate income, a is autonomous aggregate demand, i is the nominal interest rate, π is expected inflation, and m is the real money supply. All the model parameters are positive.

Set up the matrix equations when this is the structure of: a "Keynesian" model (when the endogenous variables are y and i), a "Classical" model (when the endogenous variables are m and i), and a "Post Keynesian" model (when the endogenous variables are m and y).

Now recall that we showed in class how to produce a "cookbook" inverse for any invertible 2×2 matrix. Use this "cookbook" inverse to do the matrix algebra to solve for the endogenous variables in each of the three models. Be very explicit: explicitly multiply *both* (!) sides by the inverse of the coefficient matrix, and show all your steps. After you produce your "reduced form" matrix solution for a model, also write the each reduced form equation separately (i.e., one equation for each endogenous variable) and comment on whether it is possible to determine the qualitiative effect of an increase in a. (For example, in the Keynesian model, can you say whether an increase in a leads to an increase in y?)

Exercise 13.4 (Nonlinear Comparative Statics) Consider the following IS-LM structure.

$$Y = A[i - \pi, Y, F] \qquad m = L[i, Y]$$

Using these two "structural" equations, produce a "Classical" model by selecting m and i as your endogenous variables. Following in detail the procedures introduced in class, produce the complete comparative statics for this model. Using the standard sign patterns for the partial derivatives in the structural form, try to sign the reduced form partial derivatives.

Exercise 13.5 (Two-Factor Profit Maximization) Describe the profit maximization problem for a single-good multiple-input competitive firm. Assume the firm's production function is *strictly concave*. Interpret the first-order necessary conditions for a profit maximum. Explain why a stationary point is indeed a maximum in this case.

Now consider the two-input special case, and let the production function be $\langle k, \ell\rangle \mapsto \sqrt[3]{k\ell}$. At a price of $p = 2$, with factor prices of $r = 0.50$ (for k), and $w = 0.25$ (for ℓ), what is the stationary point? Can you prove it is a maximizer?
Use a helpful 3-D plot to shed light on your results.

Exercise 13.6 (Constrained Optimization) Maximize $x^{1/3}y^{2/3}$ subject to $3x + 2y = k$. (Here k is a fixed parameter.) Proceed as follows. Set up the Lagrangian representation of this problem. Find the three first order necessary conditions. Partially reduce these three equations to eliminate the Lagrange multiplier λ, then solve for x and y using matrix algebra. How do x and y change in proportion to changes in k? (That is, what is the elasticity of x w.r.t. k, and what is the elasticity of y w.r.t. k?)

General Discussion: You may use transformations.

Illustrative Answer:
Recall that you can maximize a function $u : \mathbb{R}^n \to \mathbb{R}$ by maximizing any strictly increasing tranformation of u. Let's use that trick again to simplify the problem, noting that $\ln(x^{1/3}y^{2/3}) = (1/3)\ln x + (2/3)\ln y$. Set up the Lagrangian

$$\mathcal{L}(x, y, \lambda) = (1/3)\ln x + (2/3)\ln y - \lambda(3x + 2y - k)$$

Three FOCs:

$$
\begin{aligned}
1/3x - 3\lambda = 0 &\implies 1/9x = \lambda \\
2/3y - 2\lambda = 0 &\implies 1/3y = \lambda \\
3x + 2y - k = 0 &
\end{aligned}
$$

Using the first two equations to eliminate λ we have

$$
\begin{aligned}
3x - y &= 0 \\
3x + 2y &= k
\end{aligned}
$$

So $(x, y, \lambda) = (k/9, k/3, 1/k)$. Unit elasticities follow immediately from the proportional relationship. E.g., $(\mathrm{d}x/\mathrm{k})(k/x) = (1/9)k/(k/9) = 1$.

Our old approach was to set up a matrix solution in three variables. This time, partially reduce the system to get a system in two variables (by eliminating k). From the first two FOCs we have $y = 3x$ so we can solve

$$
\begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}
$$

which you solve in the usual way:

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ k \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix}
$$

WL Discussion: You may use a computer-algebra system (but step by step, beginning with the Lagrangian). You may find the `Eliminate` command to be useful.

Bibliography

Boughton, J. M. and E. R. Wicker (1979). The behavior of the currency-deposit ratio during the great depression. *Journal of Money, Credit and Banking 11*(4), 405–418.

Trabandt, M. and H. Uhlig (2011). The Laffer curve revisited. *Journal of Monetary Economics 58*(4), 305–327.

Previous Assignment

Homework 12

Exercise 12.1 Hoy p.383 section 9.2#1,3–6
Hoy p.383 section 9.2#7 two ways: compute $\boldsymbol{A}^3$ and then produce its determinant, and compute $\left|\boldsymbol{A}^3\right|$ and then cube it. Explain the relationship betwen these approaches. Which is more computationally efficient (and why)?
Hoy p.389 section 9.3 #1–2
Hoy section 9.4 #1,2,3, but each time, solve only for x_1.

General Discussion: Whenever Hoy provides an answer, your answer must add details and use your own words.
For 9.2.1 you may *check* your work with Mma, but explicitly use the Rule of Sarrus.
For 9.3.1 and 9.3.2, use the cofactor method, but feel free to use your cookbook method for the determinant of a 2×2 matrix. Some answers are in the Hoy appendix, but you *must* explicitly show your work.

For the section 9.4 problems, read about Cramer's rule on pp.390–393 of Hoy. This shows that you can solve the system $\boldsymbol{A}.\boldsymbol{x} = \boldsymbol{b}$ just for x_1 as the ratio of two determinants.

WL Discussion: 9.1.2: You can easily use WL to check your work. See the illustration of the use pattern matching in the previous homework assignment.
The expression `ReplacePart[A, {r_, 1} :> b[[r]] ]` replaces the first column of matrix `mA` with a conformable vector `vb`. (The trick here is pattern matching of the row indexes.)

Exercise 12.2 Hoy p.454 section 11.1 #1–7

General Discussion: Hoy offers equations that implicitly define the needed functions. Assume the x_i are function parameters. Do the problems by hand, but you can check your work with Mathematica.

WL Discussion: When you need to work with the partial-derivative defintion, you can use `Limit` to check your steps. If you want to work with expressions, use `D` to produce partial derivatives. If you prefer to work with functions, use `Derivative` to produce partial derivatives.

Exercise 12.3 Explain why a cofactor expansion by alien cofactors evaluates to 0. Using a 2×2 matrix, give an example.

Exercise 12.4 Revisit problem **??**. Solve for M using Gaussian elimination. According to your solution, what is the money multiplier? Use cofactor expansion to calculate the determinant of the coefficient matrix. Comment: you may expand across *any* column or row: choose wisely, and this will be easy. Next we will solve this system using an inverse. First find the determinant of the coefficient matrix, Next find the adjoint of the coefficient matrix, Form the inverse of the coefficient matrix as $(1/|A|)A^{\#}$, and use this to solve the system. According to your solution, what is the money multiplier?

Next solve for M using Cramer's rule. (Show all your work.) Do you get the same solution for the money multiplier?

General Discussion: Apply the specified solution methods to the full 4-equation system. Do *not* partially reduce the system before solving.

Exercise 12.5 The gradient of a function is a vector of its first-order partial derivatives. Produce the gradient for each function.

a. $\langle x, y\rangle \mapsto x^2 y$

b. $\langle x_1, x_2\rangle \mapsto x_1^2 x_2$

c. $\langle a, b\rangle \mapsto a^2 b$

d. $\langle x, y\rangle \mapsto \log[xy]$

e. $\langle x, y\rangle \mapsto \exp[xy]$

f. $\langle x, y, z\rangle \mapsto \log[xyz]$

g. $\langle x, y, z\rangle \mapsto x^2yz + xy^2z + xyz^2$

General Discussion: Function definitions are unchanged if you just change the names of the bound variables.

WL Discussion: For checking your work, possibly useful commands include `Log`, `Exp`, and `Grad`. (Use either the `D` function or `Grad` function for the differentiation.) You might also find a use for `Table`.

Computational Exercise 12.1 A differential approximation of the change in a function is a weighted sum of the partial derivatives, where the weights are the changes in the inputs.
For each function, produce differential approximation of the change in the function value at $\langle 1, 2\rangle$ when each argument increases by 10%. Compute the approximation error as the difference between the true change (Δf) and the approximate change ($\mathrm{d}f$).

a. $\langle x, y\rangle \mapsto x + 2y$

b. $\langle x, y\rangle \mapsto x^2 y$

c. $\langle x, y\rangle \mapsto x^2 y + xy^2$

d. $\langle x, y\rangle \mapsto \ln[xy]$

e. $\langle x, y\rangle \mapsto \exp[xy]$

General Discussion: Here is an example. Consdier the function $f = \langle x, y\rangle \mapsto 2xy$ At the point $\langle 1, 2\rangle$ this has the value 4. That is, $f[1, 2] = 4$. Changing each argument by 10% means the changes are $\langle \mathrm{d}x, \mathrm{d}y\rangle = \langle 0.1, 0.2\rangle$. So the actual new value is $f[1.1, 2.2] = 4.82$. The gradient is $\nabla f = \langle x, y\rangle \mapsto \langle 2y, 2x\rangle$. At the point $\langle 1, 2\rangle$ this has the value $\langle 4, 2\rangle$. That is, $\nabla f[1, 2] = \langle 4, 2\rangle$. So the differential approximation to the change is

$$\mathrm{d}f = \nabla f[x, y] \cdot \langle \mathrm{d}x, \mathrm{d}y\rangle = \langle 4, 2\rangle \cdot \langle 0.1, 0.2\rangle = 0.8$$

WL Discussion: Possibly useful commands include `Log`, `Exp`, and `Grad`. You might also find a use for `Table`.

Computational Exercise 12.2 This is the infamous oligopoly problem once again, this time with numbers. Set up for two firms a matrix equation representing the price responses of each firm firm to the other's price, as follows.

$$P_A = 5 + 0.5P_B$$
$$P_B = 5 + 0.5P_A$$

Solve for the equilibrium prices. Find the change in prices that results when firm A raises its minimum supply price by one unit.

General Discussion: Here the *minimum supply price* is the price set if the other firm's price is 0.

gnuplot Discussion: Set your <code>xrange</code> and <code>yrange</code> each to the interval from 0 to 20. (See the help for 'ranges', looking *first* at the examples.) Plot the two reaction functions. Where do they intersect? Now let marginal cost for firm A rise by 5, and add a plot representing this new reaction function. Where is the new intersection? How does this relate to your comparative statics experiment above? (Put you observations in program comments.) Make your graph look nice by moving the key and adding a title. (See the help for 'key' and 'title'.)

```
c_A=5
c_B=5
f(p) = c_A + 0.5*p
g(p) = -c_B/0.5 + p/0.5
plot [p=0:20][0:20] f(p),g(p),f(p)+5
pause -1
f(x)=x**2
dq(x0,h)=(f(x0+h)-f(x0))/h
plot [h=1:0.01] dq(2,h)
```

Exercise 12.6 (Asymmetric Tax Revenue Curve) Simple formulations of the tax-revenue curve (sometimes called the Laffer curve) are symmetric around the optimal tax rate. However, some empirical work suggests that the tax-revenue curve is not symmetric (Trabandt and Uhlig, 2011). We might represent revenues from a skewed version of curve as

$$R = \theta t \sqrt{1 - t}$$

Plot this for all tax in $[0, 1]$ given $\theta = 10$. Guess the point of maximum revenue.

Find the first-order condition. Solve for the stationary point. Check the second order condition. Draw your conclusions.

WL Discussion: You may compute answers using **WL**, but proceed step by step, using calculus fundamentals. (That is, use the necessary first-order conditions to solve for possible stationary points.) Recall that you can differentiate an expression with `D` and can differentiate a function with `Derivative`. Use `Simplify` as necessary to simplify expressions.

OPTIONAL (Ungraded) Questions

Exercise 12.7 Hoy chapter 9 RevEx #7, 8

Exercise 12.8 Consider an equation system that is represented by a list of two equations in the free variable $\boldsymbol{x}$.

$$\langle f_1[\boldsymbol{x}] = g_1[x], f_2[x] = g_2[\boldsymbol{x}] \rangle$$

The functions f_i and g_i are real-valued. For each elementary equation operation, prove that $\boldsymbol{x}^*$ solves the original system iff it solves the transformed system.

Exercise 12.9 (Partial Derivatives Review) Compute the own price, cross price, and income elasticities for the following *constant elasticity demand function* for the first of two goods: $\langle p_1, p_2, y\rangle \mapsto k_1 p_1^{\alpha_{1,1}} p_2^{\alpha_{1,2}} y^{\beta_1}$.

General Discussion: Here demand depends on $\langle p_1, p_2, y\rangle$: the price of good 1, the price of good 2, and the consumer's available income. The demand-function parameters $\alpha_{1,1}$, $\alpha_{1,2}$, and β_1 determine how sensitive demand for the first good is to these three sources. (You can focus just on the first good.)

The elasticity concepts are from intermediate microeconomics. The own price elasticity is the sensitivity of a good to its own price, e.g., if x_1 is the first good, $(\partial x_1/\partial p_1)(p_1/x_1)$ is the own price elasticity. The cross price elasticity is the sensitivity of a good to another price, e.g., if x_1 is the first good, $(\partial x_1/\partial p_2)(p_2/x_1)$ is the cross price elasticity. We usually expect the own price elasticity to be negative and the income elasticity to be positive (but less than 1), but the cross-price effect easily can be either positive or negative (substitutes vs complements). The income elasticity is of course the sensitivity of demand to income.

Exercise 12.10 (Nonlinear Comparative Statics) Two firms (A and B) produce substitutable products at constant, positive marginal cost (c_A and c_B). Each firm sets its price (P_A or P_B) as a markup over marginal cost, but the markup responds to the price set by its competitor: each raises its price (you may assume linearly) when the competitor's price rises. We can therefore represent prices setting by the two firms as $P_A = f[c_A, P_B]$ and $P_B = g[c_B, P_A]$. There is a unique equilibrium, which is at *positive* prices. (I.e., the reaction functions of the two firms cross once in $\langle P_A, P_B\rangle$-space, in the positive quadrant.)

Do the comparative statics algebra, and be sure to carefully explain the signs of the reduced form partial derivatives both intutitively and in terms of the algebra. (What is the effect on P_A of a rise in c_B? What is the effect on P_B of a rise in c_A?)
Hint: draw a graph and refer to it to when discussing your intuitive arguments. To draw this graph correctly, you must read for detail in the problem description above. Put P^A on the vertical axis and P_B on the horizontal axis. Graphically, show the effect on P_A of a rise in c_B.

WL Discussion: Mathematica provides a full suite of interactive drawing tools. **WL** also provides an extensive symbolic graphics language. The easiest approach, however, may be the use of the `ContourPlot` command.

Exercise 12.11 Consider the "Keynesian" IS-LM curve from class. (This is in my online Comparative Statics notes.)

a. Produce the total differential of the LM equation $m = L[i, Y]$ and explain how to use it to produce the slope of the LM curve.

b. Show how to write this total differential as $\boldsymbol{a} \cdot \left[\begin{smallmatrix} \mathrm{d}i \\ \mathrm{d}Y \end{smallmatrix}\right] = [\mathrm{d}m]$ by replacing $\boldsymbol{a}$ with the correct row vector.

c. Produce the total differential of the IS equation $Y = A[i - \pi, Y, F]$ and explain how to use it to produce the slope of the IS curve.

d. Show how to write this total differential as $\boldsymbol{a} \cdot \left[\begin{smallmatrix} \mathrm{d}i \\ \mathrm{d}Y \end{smallmatrix}\right] = [A_F \mathrm{d}F - A_r \mathrm{d}\pi]$ by replacing $\boldsymbol{a}$ with the correct row vector.

e. Using matrix algebra, stack these two equations to produce a single matrix equation. (For this, let the IS equation come first.)

General Discussion: Use our shortand notation for partial derivatives (e.g., write L_i instead of $\partial L/\partial i$). Learn the keyboard shortcut for entering subscripts.
Note: Use A_r to denote the partial derivative of A with respect to its first argument. (Here r is the real interest rate, i.e.,

$r = i - \pi$.)

Note: All of the details are in my online notes, but try to do this exercise without relying on them.