The Traditional Neoclassical Growth Model

This is a condensed overview of the traditional neoclassical model of exogenous long-run growth. The dynamics are in continuous time.

The neoclassical growth model of the 1950s is often called the Solow growth model or the Solow-Swan growth model. The canonical citation is [Solow-1956-QJE]_, which presents a version of the model explored below. In this model, technological change and population growth are exogenous, and the model converges to a steady state that depends on saving behavior.

Fundamental Dynamic Equation

The dynamic adjustments over time are summarized by a first-order differential equation, known as the fundamental dynamic equation of the traditional neoclassical growth model. This section derives this equation and characterizes the implied steady state.

Let the effective capital-labor ratio be k, defined as follows.

$$k = \frac{K}{L} \tag{8}$$

Here K is the capital stock and L is the effective labor supply. In this model, effective labor is not just labor hours; it includes the level of labor productivity. The variables K and L change over time, so the effective capital-labor ratio can change over time too. Apply the quotient rule of differential calculus, letting a dot indicate a time derivative. This yields the expression for the rate of change of k, as follows.

$$\dot{k} = \frac{\dot{K}}{L} - \frac{\dot{L}\,K}{L^2} \tag{9}$$

The change in the capital stock ($\dot{K}$) equals net saving. This is gross saving (S) less the depreciation of the existing capital stock ($d\,K$). The following is just an accounting relationship.

$$\dot{K} = S - d\,K$$

Next, introduce the simplifying assumption is that saving is proportional to income.

$$S = s\,Y$$

Now substitute for $\dot{K}$ and S. As a notational convenience, let g_x denote a *proportional* rate of change of x, so that $g_L \equiv \dot{L}/L$. Then we can rewrite this expression for the rate of change of the effective capital-labor ratio as follows.

$$\dot{k} = \frac{s\,Y - (d + g_L)\,K}{L} \tag{10}$$

With $y = Y/L$ and $k = K/L$, this produces the standard representation of the model dynamics.

$$\dot{k} = s\,y - (d + g_L)\,k \tag{11}$$

Here is another approach using growth-rate algebra.

$$g_k = g_K - g_L = \frac{s\,Y - d\,K}{K} - g_L = s\,y/k - (d + g_L) \to \dot{k} = s\,y - (d + g_L)\,k \tag{12}$$

Steady State

Simplifying assumptions of this growth model is that labor growth rate (g_L), the saving rate (s), and the depreciation rate (d) are constant. This introduces the possibility of a steady state, where y and k are constant. A nonzero steady state satisfies the following equations.

$$0 = s\,y - (d + g_L)\,k \tag{13}$$

$$\frac{y}{k} = \frac{d + g_L}{s} \tag{14}$$

Digression on Piketty's Second Law

[Krussell.Smith-2015-JPE]_ characterize Piketty's second law by considering the inverse of the steady-state characterization of y/k.

$$\frac{k}{y} = \frac{s}{d + g_L} \tag{15}$$

Piketty is often interpreted as saying that if g_L falls to zero, then the capital output ratio rises without bound. However, as [Krussell.Smith-2015-JPE]_ note, the equation above says only that k/y rises to s/d, where d may be approximately 5%. They suggest the following reconciliation: Piketty adopts a different theory of saving. Instead of holding constant the gross saving rate s, Piketty holds constant the net saving rate defined as $s_{\text{net}} = (S - d\,K)/(Y - d\,K)$. [Krussell.Smith-2015-JPE]_ find that this alternative characterization of saving behavior offers a worse match to the available data.

CRTS Production Technology

Model aggregate output (Y) as depending on the aggregate input of capital (K) and the aggregate input of effective labor (L), so that $Y = F[K, L]$. The neoclassical growth model assume technology displays constant returns to scale (CRTS) in it inputs. Technically, the production function is homogeneous of degree one in its inputs. By definition, this means that scaling the inputs scales the output proportionally.

$$F[\lambda\,K, \lambda\,L] \equiv \lambda\,F[K, L]$$

Incorporating CRTS

Letting $\lambda = 1/L$ produces the immediate implication that

$$F[K/L, 1] \equiv F[K, L]/L$$

With $Y = F[K, L]$, note that $Y/L = F[K/L, 1]$. Use the "per capita" notation $y = Y/L$ and $k = K/L$, and rewrite the CRTS production by defining $f[k] = F[K/L, 1]$.

$$y = f[k] \tag{16}$$

The model dynamics $\dot{k} = s\,y - (d + g_L)\,k$ become a first-order differential equation in k. This is the fundamental dynamic equation of the traditional neoclassical growth model.

$$\dot{k} = s\,f[k] - (d + g_L)\,k \tag{17}$$

A nonzero steady state is any value of k that solves the following equation.

$$\frac{f[k]}{k} = \frac{d + g_L}{s} \tag{18}$$

Exercise

Use Mathematica to derive the neoclassical growth model's fundamental dynamic equation, $\dot{k} = s\,f[k] - (d + g_L)\,k$.

A Solution:

When using WL, user defined symbols should begin with a lowercase letter; for convenient comparison to other presentations. This presentation nevertheless uses **K** and **L**.)

In[ ]:=
```
ClearAll[kdot, K, k, L, s, t, F, f, Y, y]
kdot = D[K[t]/L[t], t] (* rate of change of k=K/L *)
```

Out[ ]=
$$\frac{K'[t]}{L[t]} - \frac{K[t]\,L'[t]}{L[t]^2}$$

Start by deriving the equality $\dot{k} = s\,y - (d + g_L)\,k$. Here is one way to use WL to produce this result, using a sequence of substitution rules.

In[ ]:=
```
sub1 = K'[t] → S[t] - d K[t]; (* accounting relationship *)
sub2 = S[t] → s Y[t]; (* assume constant saving rate *)
sub3 = L'[t] → gL L[t]; (* notational convenience *)
kdot /. sub1 /. sub2 /. sub3 // Simplify
```

Out[ ]=
$$\frac{-K[t]\,(d + g_L) + s\,Y[t]}{L[t]}$$

It is conventional to rewrite this expression in "per capita" terms, where $k = K/L$ and $y = Y/L$. (This is a loose terminology; the scaling is by the *effective* labor input.)

In[ ]:=
```
sub4 = K[t] → L[t] k[t]; (* rewrite in "per capita" terms *)
sub5 = Y[t] → L[t] y[t]; (* rewrite in "per capita" terms *)
kdot = kdot /. sub1 /. sub2 /. sub3 /. {sub4, sub5} // Simplify
```

Out[ ]=
$$-k[t]\,(d + g_L) + s\,y[t]$$

This is the initial algebraic goal, slightly rearranged. Now introduce CRTS.

In[ ]:=
```
crts = λ F[K, L] == F[λ K, λ L];
crts /. {λ → 1 / L}
% /. {F[K, L] / L → y, F[k_, 1] :→ f[k]} /. K / L → k
```

Out[ ]= $\frac{F[K, L]}{L} == F\left[\frac{K}{L}, 1\right]$

Out[ ]= $y == f[k]$

```
kdot /. y → f[k] (* replace y with f[k] *)
```

Out[ ]= $-k[t]\,(d + g_L) + s\,f[k][t]$

Exercise

Use Mathematica to characterize the steady state of the neoclassical growth model.

A Solution:

Here is one way to use WL to solve for this steady-state relationship with. First, characterize the steady-state.

```
ClearAll[s, f, k, d, g] (* remove any existing definitions *)
steadyState = 0 == s f[k] - (d + gL) k;
```

Next, solve for the value of $f[k]/k$ that holds in a nonzero steady state.

In[ ]:=
```
slns = Solve[steadyState, f[k]] (* characterize steady state y *)
f[k] / k /. First@slns (* find steady-state y/k *)
```

Out[ ]= $\left\{\left\{f[k] \to \frac{k\,(d + g_L)}{s}\right\}\right\}$

Out[ ]= $\frac{d + g_L}{s}$

Exercise

[Solow-1956-QJE]_ briefly considers the effects of taxation. Suppose tax revenues are $t\,Y$, so that they are proportional to aggregate income. Correspondingly change saving behavior, so that it is proportional to disposable income, $Y - t\,Y$. How does this change the fundamental dynamic equation of the Solow model, and how does it change the steady state? Next, suppose that these tax revenues are entirely devoted to capital formation. How does this change the fundamental dynamic equation of the Solow model, and how does it change the steady state?

Marginal Products with CRTS

Under constant returns to scale, $F[\lambda\,K, \lambda\,L] == \lambda\,F[K, L]$ is an identity. For a differentiable production function, this implies that after differentiating each side with respect to λ, the two sides are still an identity.

$$F_K[\lambda K, \lambda L] K + F_L[\lambda K, \lambda L] L == F[K, L]$$

This holds true for any value of λ. (The left side is unaffected by λ because linear homogeneity of the function F implies the partial derivatives are homogeneous of degree 0.) For example, let $\lambda \to 1$. Using subscripts to indicate partial derivatives, we then get the following special case of Euler's homogeneous function theorem.

$$F_K[K, L] K + F_L[K, L] L == F[K, L]$$

Let r denote the real return to a unit of capital, often called the rental rate, and let w denote the real wage. With competitive factor markets, marginal products equal real factor prices, so that $r = F_K$ and $w = F_L$. Substitution for the partial derivatives produces the Clark-Wicksteed product exhaustion theorem, which states that factor payments exactly exhaust total production.

$$Y == r K + w L$$

Dividing both sides by L yields the equivalent statement in “per capita” terms.

$$y == r k + w$$

Next, express the marginal products F_K and F_L in terms of f instead of F. Recall that $f[k] \equiv F[k, 1]$ so that under constant returns to scale $F[K, L] = L f[K/L]$. Apply the chain rule, the product rule, and the quotient rule to produce the needed marginal products.

$$F[K,L] = L\, f[K/L]$$
$$F_K[K, L] = L f'[K/L]/L = f'[k]$$
$$F_L[K, L] = f[K/L] + L f'[K/L] * \left(-K/L^2\right) = f[k] - k f'[k]$$

Exercise

Use Mathematica to prove the Clark-Wicksteed product exhaustion theorem.

A Solution:

Mathematica labels partial derivatives with superscripts.

In[∘]:=
```
Clear[λ, F, K, L, r, w]
crts = λ F[K, L] == F[λ K, λ L]; (* definition of CRTS *)
D[crts, λ] (* differentiate both sices w.r.t. λ *)
```

Out[∘]= $F[K, L] == L F^{(0,1)}[K \lambda, L \lambda] + K F^{(1,0)}[K \lambda, L \lambda]$

Now, just set λ to 1 and substitute for the marginal products.

In[∘]:=
```
D[crts, λ] /. {λ → 1} /. {F^(1,0)[K, L] → r, F^(0,1)[K, L] → w}
```

Out[∘]= $F[K, L] == K r + L w$

Exercise

Use Mathematica to find the marginal products of K and L in terms of the “per capita” production function.

Hint:

Note that $F[K, L] == L\, f[K/L]$.

A Solution:

Use the **Grad** command to produce the vector of first-order partial derivatives (known as the *gradient*).

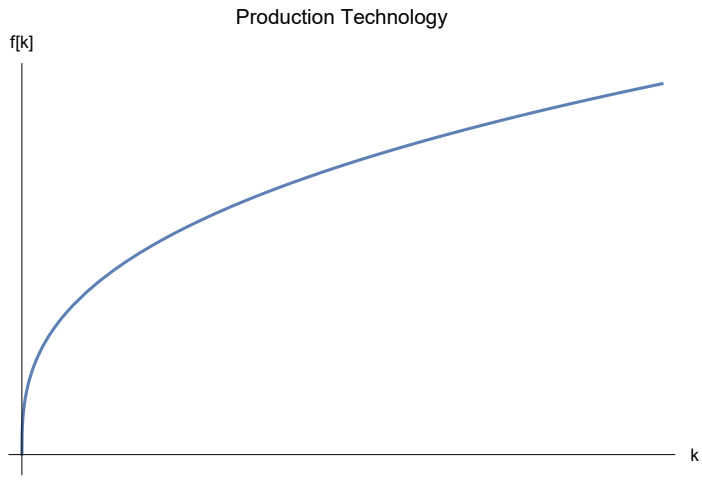
In[ ]:=
```
Grad[L * f[K / L], {K, L} ] //. {K → k * L}
```

Out[ ]=
```
{f′[k], f[k] - k f′[k]}
```

Illustrative Plots

Assume the production technology is CRTS with diminishing marginal products. A plot of the function *f* depends on the specific functional form chosen, along with a specification of the function parameters. Economists typically assume that *f*[0] == 0, that the function is continuous and concave, and that the slope initially steep but eventually flattens out.

Out[ ]=



Based on the derived expressions above, an elaboration of this simple plot can illustrate the marginal products. As a preliminary, recall that the tangent line to a real function *f* at a point k_0 can be described by a first-order Taylor expansion around k_0, as described by the following function.

```
k ⟼ f[k0] + (k - k0) f'[k0]
```

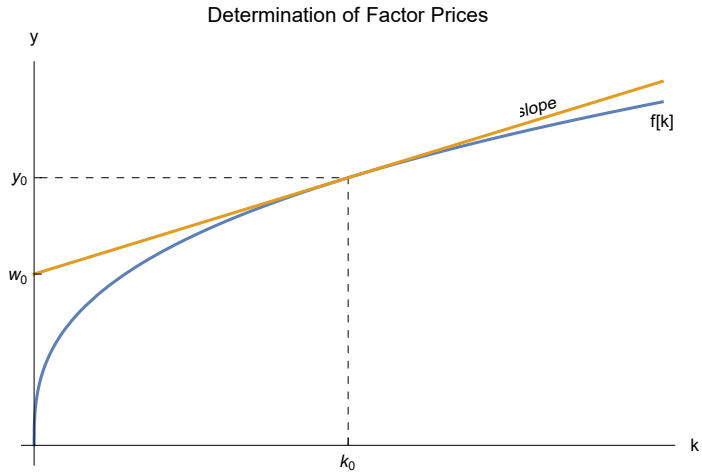
Computational Example: Tangent Function

Kicking this up a notch, produce a function that produces a tangent function. That is, for any function *f* and a point k_0, define a higher order function that returns a tangent-line to *f* at k_0. This allows us to produce a tangent to any point in the graph of *f*.

```
tangent = {f, k0} ⟼ Function[k, f[k0] + (k - k0) f'[k0]]; (* f is a parameter *)
```

Factor Prices

Out[ ]=



Cobb-Douglas Production Function

The Cobb-Douglas production function is special case of CRTS technology that is popular among economists. Production is exponential in each factor of production, with exponents that sum to unity. This section includes a constant scale factor, B, in the production function. (Increases in B are called as *Hicks-neutral* technical change.) The resulting Cobb-Douglas production function (F) can be written as follows.

$$F = \{K, L\} \longmapsto B\, K^{\alpha}\, L^{1-\alpha} \tag{19}$$

The "per capita" version (f) of this production technology becomes $k \longmapsto B\, k^{\alpha}$.

In[267]:=
```
f = k ↦ B k^α;
```

Dynamics and Steady State

Recall the fundamental dynamic equation for the neoclassical growth model under CRTS: $\dot{k} = s\, f[k] - (d + g_L)\, k$. Use the Cobb-Douglas production technology as described above to substitute for $f[k]$, so that the fundamental dynamic equation becomes

$$\dot{k} = s\, B\, k^{\alpha} - (d + g_L)\, k \tag{20}$$

Characterize a steady state by $\dot{k} = 0$.

$$0 \overset{\text{SS}}{==} s\, B\, k^{\alpha} - (d + g_L)\, k$$

With a Cobb-Douglas production technology, the equation $\dot{k}$ == 0 describes a unique positive steady-state value of k. (By inspection, we can see that k == 0 also describes a steady state.) Solve by hand for the steady-state value of k.

$$k \overset{SS}{==} \left(\frac{B\,s}{d + g_L}\right)^{\frac{1}{1-\alpha}}$$

For computational convenience, adopt a very slightly different notation.

In[131]:=
```
ss`k = (B s / (d + gL))^(1/(1-α));
```

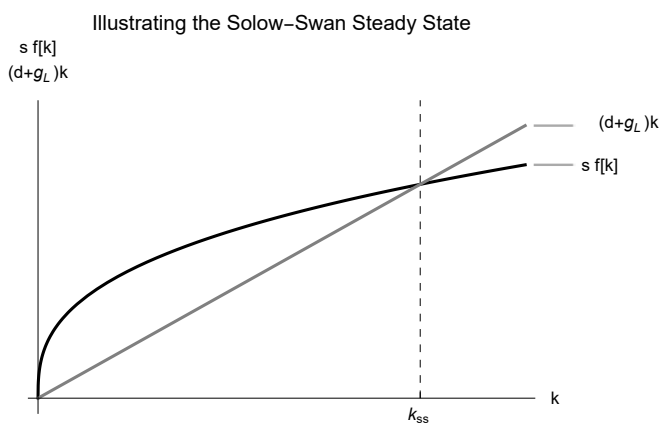
Graphical Illustration of the Steady State

The plot of the production function depends on the specific values chosen for the function parameters. Empirically, many estimates of α are around 1/3; use 0.36 for now. (A subsequent section will produce an estimate.) The value for B depends on the units chosen for K and L; for now set it rather arbitrarily to 5.0. These two values will be our first set of production parameters. Additionally, a graphical characterization of the steady state requires specification of a few more parameters: d, g_L, and s. The following values are crudely plausible in magnitude; we will experiment with them later.

In[132]:=
```
prms`cd01 = <|
   B → 5,
   α → 0.36
   |>; (* production prms chosen to match NFR *)
prms`ss20 = Join[prms`cd01, <|
    d → 0.075,
    gL → 0.01,
    s → 0.2
    |>];
```

Substituting these values leads to the following illustration of the model steady state.

Out[134]=



Exercise

Use Mathematica to solve for the steady state value of k give the Cobb-Douglas production technology.

Hint:

Use **Assuming** and **Solve**.

A Solution:

In[135]:=
```
Clear[f, B, α, s, d, n, k]
f[k_] := B k^α
ss`keq = 0 == s f[k] - k (d + gL)
```

Out[137]= $0 == -((d + gL)\, k) + B\, k^{\alpha}\, s$

Check the steady-state solution in the text with **Solve**. (The **Quiet** command silences a warning; we return to this later.)

In[138]:=
```
prms`assume = B > 0 && 1 > α > 0 && s > 0 && d > 0 && gL > 0;
soln = Assuming[prms`assume,
   Solve[ss`keq, k]
   ] // Quiet
```

Out[139]= $\left\{\{k \to 0\}, \left\{k \to \left(\frac{d + gL}{B\, s}\right)^{\frac{1}{-1+\alpha}}\right\}\right\}$

The resulting solutions are always produces as a list of lists of rules, even when there is a single solution. Use nonzero solution in this list as assign to a variable. Bind **ss`k** to the steady-state solution we have found.

In[140]:=
```
ss`k = k /. Last@soln
```

Out[140]= $\left(\frac{d + gL}{B\, s}\right)^{\frac{1}{-1+\alpha}}$

Solving with Mathematica (Solve vs. Reduce)

To produce this steady-state solution in WL, we set $\dot{k} == 0$ and then used **Solve**. There is a single solution in the resulting solution list (which ignores the trivial steady state at $k == 0$).

Here the **Solve** command gave us what we want. Nevertheless, it is informative to turn to **Reduce**, which looks for a fuller characterization of the possible solutions. Use the optional third argument to restrict the domain of possible solutions to the real numbers.

In[141]:=
```
Reduce[ss`keq && prms`assume, k, Reals] // Simplify
```

Out[141]= $\left(k == 0 \,||\, \left(\frac{d + gL}{B\, s}\right)^{\frac{1}{-1+\alpha}} == k\right) \,\&\&\, gL > 0$

Note that **Reduce** characterizes a collection of solutions as equalities, in contrast to the rules returned by **Solve**.

The trivial solution $k == 0$ is seldom of interest. Nevertheless, it is good to be reminded that it exists. It is also a good reminder that there may be more than one steady state in a dynamic model. However, in this case, the second solution yields the result we were looking for.

In sum, the non-trivial steady state of the traditional neoclassical growth model can be characterized in terms of this steady-state value of k.

Transformations Can Help Solve

Transforming an expression can help **Solve** find a solution. One approach is to ensure that a single term involves the variable whose solution is sought. For example, for $k \neq 0$, we might take advantage of the fact that $\dot{k} == 0$ iff $\dot{k}/k == 0$. The latter has a single term in k.

In[142]:=
```
Assuming[k > 0, MultiplySides[ss`keq, 1 / k]] // Simplify
Solve[%, k] // Quiet
```

Out[142]= $d + gL == B\, k^{-1+\alpha}\, s$

Out[143]= $\left\{\left\{k \to \left(\frac{d + gL}{B\, s}\right)^{\frac{1}{-1+\alpha}}\right\}\right\}$

Another salient transformation is to divide by k^{α}, which again yields a single term in k.

In[144]:=
```
Assuming[k > 0, MultiplySides[ss`keq, 1 / k^α]] // Simplify
Solve[%, k] // Quiet
```

Out[144]= $(d + gL)\, k^{1-\alpha} == B\, s$

Out[145]= $\left\{\left\{k \to \left(\frac{d + gL}{B\, s}\right)^{\frac{1}{-1+\alpha}}\right\}\right\}$

Find such transformations was important as recently as Mathematica 9, but in more recent versions they are required much less often. Nevertheless, one much approach can prove very useful: transformation of variables. Define $z = k^{1-\alpha}$. Then we can solve for the nonzero steady state by solving a linear equation in z, which is a trivial exercise for any computer algebra system. This produces the steady-state value of z, which can then be transformed in reverse to recover the steady-state value of k.

In[146]:=
```
solns`z = Solve[s B - (d + n) * z == 0, z] (* steady-state z *)
z^(1 / (1 - α)) /. First[solns`z] (* steady-state k *)
```

Out[146]= $\left\{\left\{z \to -\frac{B\, s}{-d - n}\right\}\right\}$

Out[147]= $\left(-\frac{B\, s}{-d - n}\right)^{\frac{1}{1-\alpha}}$

Consumption in the Steady State

A steady-state value of k naturally implies a steady-state value for y, since $y = f[k] = B\, k^{\alpha}$.

```
ss`y = ReplaceAll[f[k],
   k → ss`k] // Simplify (* replace k with its steady-state value *)
```

Out[554]= $B \left(\frac{B\, s}{d + gL}\right)^{\frac{\alpha}{1-\alpha}}$

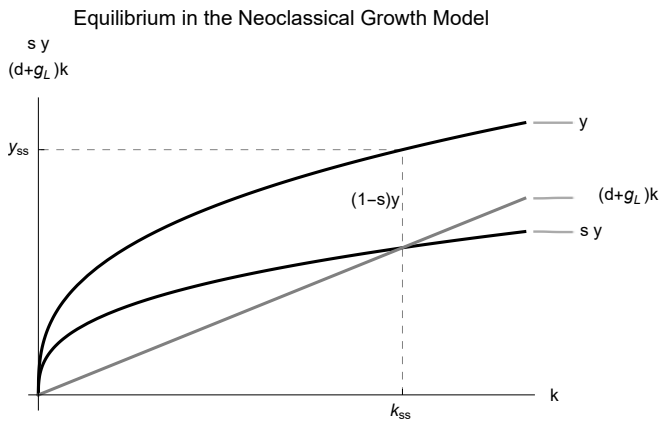
Similarly, define $c = (1 - s)\, y$, the consumption per unit of efficiency labor. Simply substitute the steady-

state value of y to produce the steady-state value of c.

In[271]:=
```
ss`c = (1 - s) ss`y
```

Out[271]= $B\,(1-s)\left(\frac{B\,s}{d+gL}\right)^{\frac{\alpha}{1-\alpha}}$

Let us look at this graphically:

Out[277]=


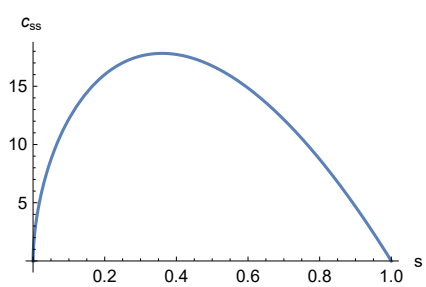
Golden Rule

This section shows that the sustainable rate of per capita consumption is not monotonically related to the saving rate. This is easy to see in the case of Cobb-Douglas production, where

In[555]:=
$$\text{ss`y} = B\left(\frac{B\,s}{d+gL}\right)^{\frac{\alpha}{1-\alpha}};$$
```
ss`c = (1 - s) ss`y;
```

This means that the effects of increase in d or g_L on c_{ss} are obvious: since they lower y_{ss}, they must lower c_{ss}. However, an increase in s is harder to analyze: it increases y_{ss}, but it reduces the proportion of y_{ss} consumed. An increase in s may either raise or lower consumption. On the one hand, more saving means less consumption out of any give level of income. On the other hand, more saving means more capital accumulation and thereby higher income.

To illustrate this, note that if s == 0 then there is no consumption in the steady state. (Capital that depreciates is not replaced.) And of course if s == 1, then there is no consumption. So we expect that steady-state consumption initially rises in s but eventually falls.

Out[698]=


Cobb-Douglas Algebra

Naturally, we would like to characterize the top of this curve: the value of s that maximizes steady-state c. This is called the *golden rule* saving rate.

The *golden rule of capital accumulation* sets steady-state consumption as high as possible. Recall that the Cobb-Douglas functional form implies that steady-state consumption is $B\,(1-s)\left(\frac{B\,s}{d+gL}\right)^{\frac{\alpha}{1-\alpha}}$, which makes it easy to solve for the optimal saving rate. First, use the chain rule and the power rule to differentiate c_{ss} with respect to s.

In[700]:=
```
D[ss`c, s] // Simplify (* find the slope *)
```

Out[700]=
$$\frac{B\left(\frac{B\,s}{d+gL}\right)^{\frac{\alpha}{1-\alpha}}(s-\alpha)}{s\,(-1+\alpha)}$$

This slope is 0 at the maximum, so produce the necessary first order condition and then solve for s.

In[694]:=
```
Solve[0 == D[ss`c, s], s] (* solve FOC for s *)
```

Out[694]= $\{\{s \to \alpha\}\}$

How saving should change depends on whether α is greater than or smaller than s. Note that an economy can save too much! If saving exceeds the golden rule, more saving will raise the capital stock and thus income, but it will still lower consumption. But when saving less than the golden rule, more saving will mean reduced consumption in the short-run but will raise consumption in the long run. Finally, when saving less than the golden rule, if we further reduce our saving, there will be a short-run increase in consumption. But eventually we pay for it with permanently lower consumption.

Maximum sustainable consumption requires $s == \alpha$. This is the golden rule of saving.

Algebra for the General Case

Approaching the golden rule more abstractly may offer a gain in intuition. Consider picking s to maximize $s\,f[k]$, subject to the steady-state constraint that $s\,f[k] == (d+g_L)\,k$.

$$\begin{aligned} &\max_{s,k}\ (1-s)\,f[k] \\ &s.t.\ s\,f[k] = (d+g_L)\,k \end{aligned} \tag{21}$$

Note how the constraint restricts the covariation in s and k (since the steady state k depends on s). Set up the Lagrangian for this constrained optimization, and then produce first-order necessary conditions from the gradient. (Assume the constraint is binding.)

In[598]:=
```
ℒ = (1 - s) f[k] + λ (s f[k] - (d + gL) k);
```

Using this Lagrangian, find the optimal s and k in the usual way: set the gradient to 0 and solve the resulting system.

In[779]:=
```
gradient = Grad[ℒ, {s, k, λ}];
gr`focs = Thread[gradient == 0]
```

Out[780]= $\{-f[k] + \lambda f[k] == 0,\ (1-s) f'[k] + \lambda (-d - gL + s f'[k]) == 0,\ -((d + gL) k) + s f[k] == 0\}$

Out[781]=

	first-order conditions
s	$-f[k] + \lambda f[k] == 0$
k	$(1-s) f'[k] + \lambda (-d - gL + s f'[k]) == 0$
λ	$-((d + gL) k) + s f[k] == 0$

First, since the constraint binds, $\lambda \neq 0$, and the first equation therefore gives us $\lambda == 1$. Since $\lambda == 1$, second equation then tells us that $f'[k] = n + g_L$, and the third just reiterates our constraint.

We can therefore find k such that the slope of the production function matches the slope of the depreciation curve, and then choose s to make that level of k the steady state level. That is, first find k so that $f'[k] = d + g_L$, and then find s such that $s f[k] = k f'[k]$. (So the golden-rule s equals the k-elasticity of f.)

Recall that in the particular case of our Cobb-Douglas function, this elasticity is α.

In[606]:= $\text{gr`s} = \frac{k f'[k]}{f[k]}$ /. $\{f[k] \to B k^{\alpha},\ f'[k] \to \alpha B k^{\alpha-1}\}$

Out[606]= α

Similarly, a Cobb-Douglas production technology allows us to solve for the golden-rule level of the capital stock and income.

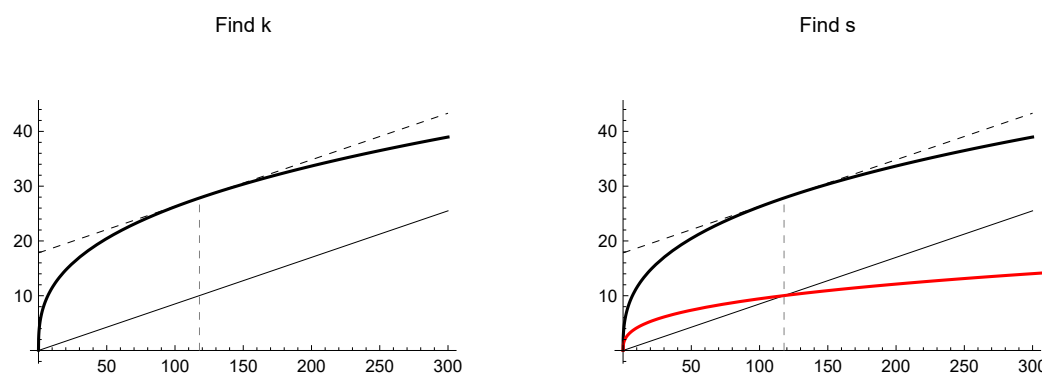
In[621]:=
```
gr`k = ss`k /. {s → α}
gr`y = ss`y /. {s → α}
```

Out[621]= $\left(\frac{B\alpha}{d + gL}\right)^{\frac{1}{1-\alpha}}$

Out[622]= $B\left(\frac{B\alpha}{d + gL}\right)^{\frac{\alpha}{1-\alpha}}$

If we specify numerical values for the parameters, we can illustrate the solution process graphically. Just for the sake of the graphical display, our plots will use a much higher saving rate.

Out[713]=



Golden Rule

Another Approach

Begin with a logarithmic transformation of c_{ss}.

In[203]:=
```
Log[ss`c]
% // PowerExpand
D[%, s]
Solve[0 == %, s]
```

Out[203]= $\text{Log}\left[B\,(1-s)\left(\frac{B\,s}{d+gL}\right)^{\frac{\alpha}{1-\alpha}}\right]$

Out[204]= $\text{Log}[B]+\text{Log}[1-s]+\frac{\alpha\,(\text{Log}[B]-\text{Log}[d+gL]+\text{Log}[s])}{1-\alpha}$

Out[205]= $-\frac{1}{1-s}+\frac{\alpha}{s\,(1-\alpha)}$

Out[206]= $\{\{s\to\alpha\}\}$

A More Traditional Presentation

Exercise

Steady-State Predictions of the Model

In order to explore the predictions of the neoclassical growth model, distinguish between the labor input N and the effective labor input $L = A\,N$. (Here, changes in A are labor-augmenting or *Harrod-neutral* technical change.) Relate their growth rates using standard growth algebra. Two common notations for the growth rate of x are $\hat{x}$ and g_x. When convenient, we will additionally write $g[x]$.

$$\hat{L} = \hat{A} + \hat{N} \tag{22}$$

$$g_L = g_A + g_N \tag{23}$$

Correspondingly, the fundamental dynamic equation is often written as:

$$\dot{k} = s\,f[k] - (d + g_A + g_N)\,k \tag{24}$$

When production is Cobb-Douglas, the associated steady-state capital stock is $k_{ss} = \left(\frac{B\,s}{d+g_A+g_N}\right)^{1/(1-\alpha)}$.

The neoclassical growth model typically treats g_A and g_N as exogenous constants. Since $y = Y/L = Y/(A\,N)$ is constant in the steady state, technological change implies that Y/N must be growing constantly. This is a core prediction of the traditional neoclassical growth model.

Exercise

Use Mathematica to show that the growth rate of a product is a sum of growth rates.

Hint:

The **Dt** command produces a total derivative.

A Solution:

First show that the growth rate of a variable is the rate of change of its logarithm.

In[154]:=
```
Dt@Log[L]
```

Out[154]= $\frac{\text{Dt}[L]}{L}$

Use this fact to compute the growth rate of a product.

In[155]:=
```
With[{L = A * N}, Dt@Log[L]] // Apart
```

Out[155]= $\frac{\text{Dt}[A]}{A} + \frac{\text{Dt}[N]}{N}$

Stylized Facts

Kaldor (1957, 1961) offered six famous *stylized facts* about the growth of advanced industrialized economies. In this subsection, the function *g* yields the proportional rate of change of its argument. Here *N* is the input of labor hours (not effective labor).

- $g[Y/N] > 0$ and relatively constant (i.e., exponential growth in living standards, as measured by income per worker)
- $g[K/N] > 0$ and relatively constant (i.e., stable growth in capital-labor ratio)
- $g[Y/K] \approx 0$ (i.e., fairly stable capital-output ratio)
- $g[\Pi/K] \approx 0$ (i.e., no trend in the rate of return to capital) given the investment share, which is correlated with profit share
- $g[Y]$ and $g[Y/N]$ vary a lot across countries
- income shares are relatively constant, with high Π/Y correlated with high I/Y

In the steady state, the overall standard of living keeps up with technological change.

$$\begin{aligned} 0 = \hat{y} = g[Y/(AN)] &= g[Y/N] - g[A] \\ g[Y/N] &= g[A] \end{aligned} \tag{25}$$

Similarly, in a steady state

$$\begin{aligned} 0 = g[K/(AN)] &= g[K/N] - g[A] \\ g[K/N] &= g[A] \end{aligned} \tag{26}$$

Note that the steady-state capital-output ratio will have to be constant since *y* and *k* are constant:

$$Y/K == \frac{Y}{AN} \Big/ \frac{K}{AN} == y/k \tag{27}$$

A steady state has a trendless capital-output ratio along with a trendless rate of return to capital. These imply the shares of capital and labor will be roughly constant. The profit rate is constant, and

the capital output ratio is constant, so correspondingly profits are a constant share of output (since the capital-output ratio is constant.)

$$\Pi == r\,K == f'[k_{ss}]\,K \tag{28}$$

$$\frac{\Pi}{K} == \frac{\Pi}{Y}\,\frac{Y}{K} \tag{29}$$

Kaldor's fifth stylized is relatively constant income shares, with a high profit share being correlated with high investment share. This also requires more discussion. It says that, across steady states, we should see a high profit share when we have a high saving rate. However, give a Cobb-Douglas production technology and competitive factor markets, the profit share is determined by the capital-elasticity of output (α).

$$\frac{\Pi}{Y} == \frac{r\,K}{F[K,\,L]} == \frac{r\,k}{f[k]} == f'[k]\,\frac{k}{f[k]} == \alpha \tag{30}$$

(Make sure you can prove the last step.)

The last stylized fact is that labor productivity and output growth vary a lot across countries. This is more problematic. We can get any variation we wish in Y/N by varying s and n, but varying the growth rate requires more discussion. We will return to this.

Steady-State Comparative Statics

Recall solution for the steady-state capital-stock, given our Cobb-Douglas technology.

In[816]:=
$$\text{ss`k} = \left(\frac{B\,s}{d + gL}\right)^{\frac{1}{1-\alpha}};$$

First consider the responses of the term in parentheses.

In[817]:=
$$\text{Grad}\left[\frac{B\,s}{d + gL},\ \{s,\ d,\ gL,\ B\}\right]$$

Out[817]=
$$\left\{\frac{B}{d + gL},\ -\frac{B\,s}{(d + gL)^2},\ -\frac{B\,s}{(d + gL)^2},\ \frac{s}{d + gL}\right\}$$

This makes it easy to understand the response of the steady-state k, by application of the chain rule.

In[818]:=
```
Grad[ss`k, {s, d, gL, B}]
```

Out[818]=
$$\left\{\frac{B\left(\frac{B\,s}{d+gL}\right)^{-1+\frac{1}{1-\alpha}}}{(d + gL)\,(1 - \alpha)},\ -\frac{B\,s\left(\frac{B\,s}{d+gL}\right)^{-1+\frac{1}{1-\alpha}}}{(d + gL)^2\,(1 - \alpha)},\ -\frac{B\,s\left(\frac{B\,s}{d+gL}\right)^{-1+\frac{1}{1-\alpha}}}{(d + gL)^2\,(1 - \alpha)},\ \frac{s\left(\frac{B\,s}{d+gL}\right)^{-1+\frac{1}{1-\alpha}}}{(d + gL)\,(1 - \alpha)}\right\}$$

Another perspective arises if we simplify this expression:

In[819]:=
```
ss`exogs = {s, d, gL, B};
dks = Grad[ss`k, {s, d, gL, B}] // Simplify
```

Out[820]=
$$\left\{-\frac{\left(\frac{B\,s}{d+gL}\right)^{\frac{1}{1-\alpha}}}{s\,(-1 + \alpha)},\ \frac{\left(\frac{B\,s}{d+gL}\right)^{\frac{1}{1-\alpha}}}{(d + gL)\,(-1 + \alpha)},\ \frac{\left(\frac{B\,s}{d+gL}\right)^{\frac{1}{1-\alpha}}}{(d + gL)\,(-1 + \alpha)},\ -\frac{\left(\frac{B\,s}{d+gL}\right)^{\frac{1}{1-\alpha}}}{B\,(-1 + \alpha)}\right\}$$

Now we see the steady-state capital stock in each expression. Substitute to make this explicit.

In[821]:=
```
dks /. {(B s/(d + gL)) → kss^(1 - α)} // Simplify
```

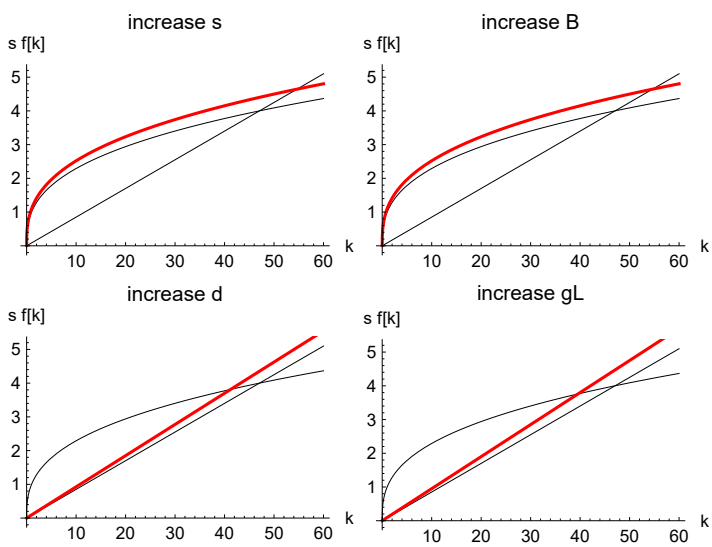
Out[821]= $\left\{\frac{kss}{s - s\,\alpha}, \frac{kss}{(d + gL)\,(-1 + \alpha)}, \frac{kss}{(d + gL)\,(-1 + \alpha)}, \frac{kss}{B - B\,\alpha}\right\}$

Rewrite this as follows, which is readily signed as {+, −, −, +}.

$$\frac{\partial k_{ss}}{\partial\{s,d,g_L,B\}} = \left\{\frac{k_{ss}}{s\,(1-\alpha)}, \frac{-k_{ss}}{(d+g_L)\,(1-\alpha)}, \frac{-k_{ss}}{(d+g_L)\,(1-\alpha)}, \frac{k_{ss}}{B\,(1-\alpha)}\right\} \tag{31}$$

Graphical illustration of these comparative static results:

Out[824]=



Since $y = f[k]$, we expect exactly the same qualitative comparative statics for y. Specifically, for parameters that do not change f, $dy_{ss} = f'[k_{ss}]\,dk_{ss}$. Since $f' > 0$, the change in y_{ss} is in the same direction as the change in k_{ss}. The following algebra confirms this. (Note that B is a production function parameter, so we leave it out here for simplicity.)

In[838]:=
```
With[{f = k ↦ B k^α},
  ss`y = f[ss`k]
 ] // Simplify
```

Out[838]= $B\left(\frac{B\,s}{d + gL}\right)^{\frac{\alpha}{1-\alpha}}$

In[839]:=
```
dys = Grad[ss`y, {s, d, gL}] // Simplify
(* check that the chain rule produces the same result: *)
dys == B * α * ss`k^(α - 1) * Grad[ss`k, {s, d, gL}] // Simplify
```

Out[839]= $\left\{-\frac{B\left(\frac{B\,s}{d+gL}\right)^{\frac{\alpha}{1-\alpha}}\alpha}{s\,(-1+\alpha)}, \frac{\left(\frac{B\,s}{d+gL}\right)^{\frac{1}{1-\alpha}}\alpha}{s\,(-1+\alpha)}, \frac{\left(\frac{B\,s}{d+gL}\right)^{\frac{1}{1-\alpha}}\alpha}{s\,(-1+\alpha)}\right\}$

Out[840]= True

It should be no surprise that once again, we notice k_{ss} everywhere in this expression. Make this explicit by substitution.

In[841]:=
```
dys /. {(B s)/(d + gL) → kss^(1 - α)} // Simplify
```

Out[841]=
$$\left\{\frac{B\,kss^{\alpha}\,\alpha}{s-s\,\alpha},\ \frac{kss\,\alpha}{s\,(-1+\alpha)},\ \frac{kss\,\alpha}{s\,(-1+\alpha)}\right\}$$

By inspection, we can simplify even further.

In[842]:=
```
dys == α/(s (1 - α)) {ss`y, -ss`k, -ss`k} // Simplify
```

Out[842]= True

Now, do all of these comparative statics at one go.

In[843]:=
```
Grad[{ss`k, ss`y}, {s, d, gL}] // Simplify;
ss`dkydsdn = % /. {(B s)/(d + gL) → kss^(1 - α)} // Simplify
TableForm[ss`dkydsdn,
 TableHeadings → {{k, y}, {s, d, gL}}]
```

Out[844]=
$$\left\{\left\{\frac{kss}{s-s\,\alpha},\ \frac{kss}{(d+gL)\,(-1+\alpha)},\ \frac{kss}{(d+gL)\,(-1+\alpha)}\right\},\ \left\{\frac{B\,kss^{\alpha}\,\alpha}{s-s\,\alpha},\ \frac{kss\,\alpha}{s\,(-1+\alpha)},\ \frac{kss\,\alpha}{s\,(-1+\alpha)}\right\}\right\}$$

Out[845]//TableForm=

	s	d	gL
k	$\frac{kss}{s-s\,\alpha}$	$\frac{kss}{(d+gL)\,(-1+\alpha)}$	$\frac{kss}{(d+gL)\,(-1+\alpha)}$
y	$\frac{B\,kss^{\alpha}\,\alpha}{s-s\,\alpha}$	$\frac{kss\,\alpha}{s\,(-1+\alpha)}$	$\frac{kss\,\alpha}{s\,(-1+\alpha)}$

Exercise: explain why dyss/dk ≠ df[*k*]/dk. (Hint: compare each element, and review the definition of *y*[].)

In[833]:=
```
TableForm[Simplify[Sign[ss`dkydsdn], kss > 0 && α > 0],
 TableHeadings → {{k, y, c}, {s, d, gL}}, TableAlignments → Center]
```

Out[833]//TableForm=

	s	d	gL
k	1	-1	-1
y	1	-1	-1

The MRW Data

This section provides some descriptive insights into the dataset provided by [Mankiw.Romer.Weil-1992-QJE]_ (MRW) in an appendix to their paper. The data were imported by optical scan from a PDF of this data appendix. The replication results reported below are quite close but not perfect, so either the scan is not perfect or the appendix contains a typo.

Create the Datasets

The MRW data comes from the real national accounts of [Summers.Heston-1988]_, covering 1960-1985. GDP per working-age adult is provided for 1960 and 1985, and g_N is the average growth 1960-85 of this 15–64 age group (“working age”). In addition, as discussed below, they impose $g_A + \delta == 0.05$. (They claim plausible changes do not affect the results.)

The saving rate is measured as the investment proportion of GDP ($s = I/GDP$). A dataset subtlety is that, since investment rates are not constant over time, MRW average them over the period. Using dummy variables, these countries have been classified as belong to three groups: N, I, and O denote the non-oil, intermediate, and OECD samples.

In[339]:=
```
mrwDataLocation = "https://subversion.american.edu/aisaac/hw/data/mrw1992.csv";
(* online data *)
(* mrwDataLocation=dataFolder<>"mrw1992.csv"; *) (* for downloaded data *)
SetOptions[Dataset, MaxItems → 5]; (* limit size of display *)
dsAll = SemanticImport[mrwDataLocation] (* create a dataset *)
```

Out[340]=

Country	N	I	O	Y1960	Y1985	GDP	agepop	IoY
Zimbabwe	1	1	0	1187	2107	5.1	2.8	21.1
Afghanistan	0	0	0	1224	—	1.6	—	6.9
Bahrain	0	0	0	—	—	—	—	30.0
Bangladesh	1	1	0	846	1221	4.0	2.6	6.8
Myanmar	1	1	0	517	1031	4.5	1.7	11.4

rows 43–47 of **121**

It seems that they have data on 121 countries.

In[304]:=
```
Length[dsAll]
```

Out[304]= 121

However, scrolling through the data reveals quite a few missing values. To get a sense of how pervasive this is, create a query. This dataset is essentially a list of associations, where each association maps field names (as displayed in the header) to values. Use the **MemberQ** command to find out if any of the values of an association are missing (i.e., have the head **Missing**). Use the **CountsBy** command to query the dataset about how many records have fields with missing values. Missing values have a head of **Missing**, so the **MemberQ** command can look for missing field values.

In[341]:=
```
Query[CountsBy[MemberQ[_Missing]]]@dsAll
```

Out[341]=

False	104
True	17

Data Wrangling

Filter out such records with the **DeleteMissing** command. Take a little care to read the documentation, because the goal is do delete the entire record whenever any of its values are **Missing**. As expected from our previous exploration, the new dataset contains 104 records.

In[342]:=
```
dsNoMissing = DeleteMissing[dsAll, 1, 1];
Length@dsNoMissing
```

Out[343]= 104

Next, construct the three subsets considered by MRW by selecting based on their dummy variables. This is called subsetting, selective set building, or filtering.

Look up the **Function** command, you will find that is has an ampersand postfix shorthand, which is commonly used by WL programmers. When a field name is a string, WL allows you to prefix it with an octothorpe (#) in order to refer to it in a function. This makes it easy to subset the dataset based on the provided value of a dummy variable. The following **Select** commands each have a function argument that uses this notation.

In[349]:=
```
dsN = Select[#N == 1 &]@dsNoMissing; (* nonoil *)
dsI = Select[#I == 1 &]@dsNoMissing; (* intermediate (subset of dsN) *)
dsO = Select[#O == 1 &]@dsNoMissing; (* 22 high pop OECD countries (subset of dsI) *)
```

As a check on these subsets, use the **Length** command to count the number of observations in each dataset. (The **/@** operator is an infix shorthand for the **Map** command, which applies a function to each element of a list.)

In[778]:=
```
Map[Length, {dsN, dsI, dsO}]
```

Out[778]= {98, 75, 22}

Note that the intermediate countries are a subset of the non-oil countries.

In[354]:=
```
Select[1 == #N && 1 == #I &]@dsNoMissing
```

Out[354]=

Country	N	I	O	Y1960	Y1985	GDP	agepop	IoY
Algeria	1	1	0	2485	4371	4.8	2.6	24.1
Botswana	1	1	0	959	3671	8.6	3.2	28.3
Cameroon	1	1	0	889	2190	5.7	2.1	12.8
Ethiopia	1	1	0	533	608	2.8	2.3	5.4
Ivory Coast	1	1	0	1386	1704	5.1	4.3	12.4

rows 1–5 of 75

Countries whose GDP is primarily from oil production are not in any of these groups. For example, Kuwait is not.

In[363]:=
```
Select[dsNoMissing, #Y1960 > 50 000 &]
```

Out[363]=

Country	N	I	O	Y1960	Y1985	GDP	agepop	IoY
Kuwait	0	0	0	77 881	25 635	2.4	6.8	9.5

Examine the Data

The Cross-Country Distribution of Income

The cross-country distribution of income has shifted substantially over time. On a cross-country basis, it seems that the mean income has risen while the variance has fallen.

In[441]:=
```
dsNoMissing[Mean, {"Y1960", "Y1985"}]
```

Out[441]=

Y1960	3734.35
Y1985	5547.19

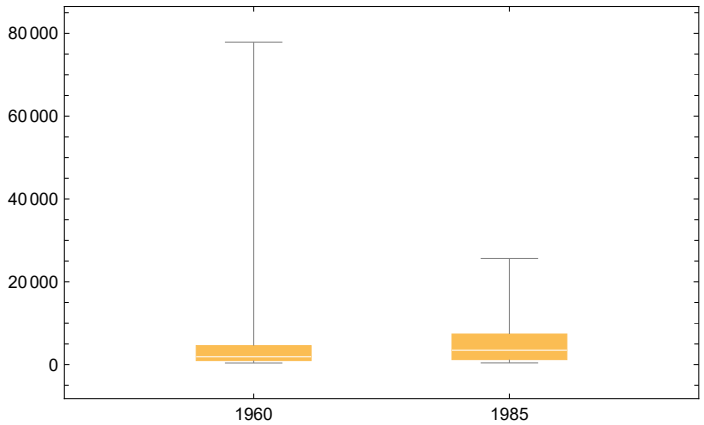
In[442]:=
```
dsNoMissing[Variance, {"Y1960", "Y1985"}]
```

Out[442]=

Y1960	61 876 981.
Y1985	30 686 580.

In[398]:=
```
BoxWhiskerChart[
 Transpose@dsNoMissing[All, {"Y1960", "Y1985"}],
 ChartLabels → {"1960", "1985"}]
```

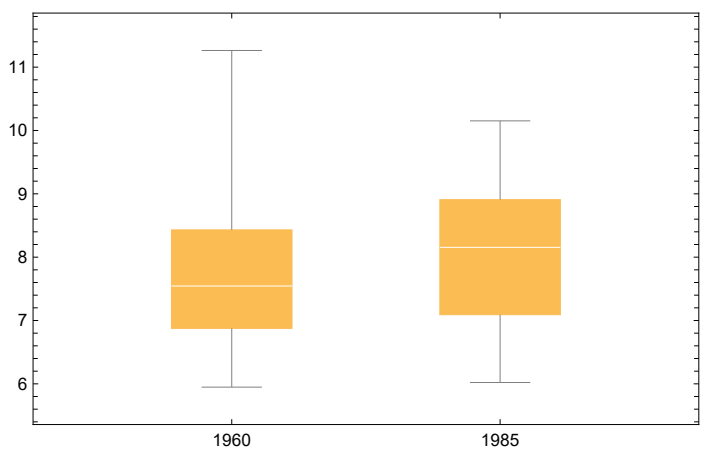
Out[398]=



Check whether the logarithms appear more normally distributed.

In[401]:=
```
BoxWhiskerChart[
 Transpose@dsNoMissing[All, {"Y1960", "Y1985"}, Log],
 ChartLabels → {"1960", "1985"}]
```

Out[401]=



Economists often use histograms to provide a more detailed look at such distributions. Notice the substantial flattening and rightward shift of the distribution.