1 Classical Model of the Price Level

Rational expectations hypothesis: the expectations relevant to economic outcomes are appropriately proxied by the forecasts derived from the economist’s model.

Consider the log-linear representation of the price level, as determined by the Classical model.

\[ p_t = m_t - \phi y_t + \lambda i_t \]  

(1)

As always, the Classical model tells us that the price level is determined by nominal money supplies and real money demands. Together these two components yield our crude “Classical” model of the price level.

One potential problem for empirical tests such a crude “Classical” model is that it may mistake other influences on the price level for the response to the interest rate. For example, the empirical version of the model allows for random shocks to money demand. Let \( u_t \) be the money demand shock at time \( t \), so that

\[ p_t = m_t - \phi y_t + \lambda i_t - u_t \]  

(2)

The money demand shocks (\( u_t \)) affect the current price level, but they also influence expected future inflation and thus the interest rate.

For example, suppose \( u_t \) is completely temporary (no serial correlation). Then a positive shock will increase money demand and lower the price level today, but next period the shock will be absent and the price level will rise again. So a positive money demand shock may have both a direct and indirect effect on the price level: by raising money demand it decreases the price level. But it may also raise expected future inflation (and the interest rate), thereby increasing the price level.

Furthermore, since the observable interest rate will be positively correlated with the unobserved money demand shocks, the estimate of \( \lambda \) will be biased downward. (This was noted by Sargent (1977 IER).)

2 Expectations and the Price Level

In this handout, we algebraically derive the predictions of the “Classical” model of price determination under the rational expectations hypothesis. This allows us to overcome some of the problems that have concerned us, although it raises a few new questions as well.

We begin by recalling the Fisher effect.

\[ i_t = r_t + \Pi_t^e \]  

(3)

We use the resulting expression to substitute for the interest rate in our crude “Classical” model.

\[ p_t = m_t - \phi y_t + \lambda \Pi_t^e + \lambda r_t - u_t \]  

(4)

2.1 Fundamentals

Let us combine all the exogenous determinant of the price level into a single variable, \( \tilde{m} \).

\[ \tilde{m}_t = m_t - \phi y_t + \lambda r_t - u_t \]  

(5)
We will refer to $\tilde{m}$ as the price level fundamentals. This allows us to rewrite (4) in a slightly simpler form.

$$p_t = \tilde{m}_t + \lambda \Pi_t$$  \hspace{1cm} (6)

Equation (6) expresses the price level in terms of the price-level fundamentals and expected inflation. We see that an increase in expected inflation causes the price level to rise: expectations are a crucial determinant of the price level. The price level is determined by the price level fundamentals and expectations. Unless $\lambda = 0$, the fundamentals alone are not enough to determine the price level.

Note that if no inflation is expected, then price-level fundamentals directly determine the price level. Changes in the price level are very hard to predict, and when we have no reason to believe that inflation or deflation is more likely, our best guess may well be that it will remain unchanged. (That does not mean that we think the price level will not change; it is just the guess that is best on average no matter how it actually does change.) In such cases we may say, roughly, that the price level is believed to follow a random walk: each period it is just as likely to rise as to fall, and we have no reason to bet on a movement in one direction rather than another.

This situation presents obvious difficulties for empirical work: expectations are not observable. There are many proposals to deal with this, but we will only mention two. First, we might ask market participants what their expectations are. That is, we might rely on surveys. Although such suggestions seem to have recently gained some ground, traditionally economists have rejected this approach. Instead, there has been a tendency to finesse the observability problem by modeling expectations formation. At this point, most economists turn to the rational expectations hypothesis.

$\Pi_t$ represents the expectation at time $t$ of the percentage rate of inflation of the price level over the period $t$ to $t+1$. Expected inflation can be decomposed into the current price level and the expected future price level.

$$\Pi_t = p_{t+1}^e - p_t$$  \hspace{1cm} (7)

We can therefore write

$$p_t = \tilde{m}_t + \lambda(p_{t+1}^e - p_t)$$  \hspace{1cm} (8)

Solving for the price level yields

$$p_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} p_{t+1}^e$$  \hspace{1cm} (9)

Once again this solution drives home a key message: the current price level depends on its expected future value.

3 Rational Expectations

Rational expectations hypothesis: the expectations relevant to economic outcomes are appropriately proxied by the forecasts derived from the economist’s model.

This section contains an algebraic presentation of the “Classical” to price level determination under rational expectations. The rational expectations hypothesis (REH) is that the expectations relevant to economic outcomes are appropriately proxied by the forecasts derived from the economist’s model. Let $E_t$ denote a mathematical expectation conditional on all information available at time $t$ (including past values of the fundamentals and the structure of the model). Then for the “Classical” model, we will represent the rational expectation hypothesis by

$$p_{t+1}^e = E_t p_{t+1}$$  \hspace{1cm} (10)

The REH renders the expected future price level one of the variables explained by the model. This is an apparently simple way of endogenizing the expectations of the future, but it has very strong implications. We will show that it implies that the entire expected future of the economy is relevant to the current price level. Specifically, as noted by, it implies that the current price level is a weighted sum of all expected future price-level fundamentals:

$$p_t = \frac{1}{1 + \lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^i E_t \tilde{m}_{t+i}$$  \hspace{1cm} (11)
Since the weights decline over time, just as in a discounted present value calculation, this is often referred to as the “present value” solution for the price level.

Equation (11) is the solution for the price level under rational expectations. Since it involves expectations of future fundamentals, it is not yet clear how we can use this solution in empirical work. Before addressing that question, you may want to work through the rational expectations algebra. (Otherwise, skip to the next section.)

### 3.1 The Rational Expectations Algebra

This section derives the price level solution (11).

Recall our solution for the “Classical” model:

\[
 p_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} p_{t+1} 
\]  

(9)

We will combine this with our rational expectations hypothesis:

\[
 p_{t+1} \equiv \mathcal{E}_t p_{t+2} 
\]  

(12)

The result is

\[
 p_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t p_{t+2}
\]  

(13)

We are going to use (13) to express the price level in terms of fundamentals. Our first approach will use “recursive substitution.” We begin the observation that (13) applies at each point in time. Therefore it implies

\[
 p_{t+1} = \frac{1}{1 + \lambda} \tilde{m}_{t+1} + \frac{\lambda}{1 + \lambda} \mathcal{E}_{t+1} p_{t+2}
\]  

(14)

This is just the same relationship, one period forward in time. Taking expectations (at time \( t \)) of both sides of this yields

\[
 \mathcal{E}_t p_{t+1} = \mathcal{E}_t \left\{ \frac{1}{1 + \lambda} \tilde{m}_{t+1} + \frac{\lambda}{1 + \lambda} \mathcal{E}_{t+1} p_{t+2} \right\}
\]

\[
 = \frac{1}{1 + \lambda} \mathcal{E}_t \tilde{m}_{t+1} + \frac{\lambda}{1 + \lambda} \mathcal{E}_t \mathcal{E}_{t+1} p_{t+2}
\]

(15)

Of course, now we need to substitute for \( \mathcal{E}_{t+1} p_{t+2} \), and then \( \mathcal{E}_{t+2} p_{t+3} \) etc. Then we have after \( n \) substitutions we have

\[
 p_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t \left\{ \frac{1}{1 + \lambda} \tilde{m}_{t+1} \right\}
\]

\[
 + \cdots + \left( \frac{\lambda}{1 + \lambda} \right)^n \mathcal{E}_t \mathcal{E}_{t+1} \cdots \mathcal{E}_{t+n-1} \left\{ \frac{1}{1 + \lambda} \tilde{m}_{t+n} \right\}
\]

\[
 + \left( \frac{\lambda}{1 + \lambda} \right)^{n+1} \mathcal{E}_t \mathcal{E}_{t+1} \cdots \mathcal{E}_{t+n} p_{t+n+1}
\]

(16)

The Law of Iterated Expectations says that \( \mathcal{E}_t \{ \mathcal{E}_{t+i} p_{t+i+1} \} = \mathcal{E}_t p_{t+i+1} \). This just means that your current best guess of your future best guess about the future price level is just your current best guess about that future price level. In other words, your current guess uses all the information you have available, and therefore differs from your future best guess only due to new information you will receive in the future and cannot use now. This allows us to simplify our last solution:

\[
 p_t = \frac{1}{1 + \lambda} \sum_{i=0}^{n} \left( \frac{\lambda}{1 + \lambda} \right)^i \mathcal{E}_t \tilde{m}_{t+i} + \left( \frac{\lambda}{1 + \lambda} \right)^{n+1} \mathcal{E}_t p_{t+n+1}
\]

(18)
Noting that \( \frac{\lambda}{1 + \lambda} < 1 \), we often assume \( \lim_{n \to \infty} \frac{\lambda}{(1 + \lambda)^n} \mathcal{E}_t p_{t+n+1} = 0 \). (For now we make this assumption: the weighted sum of expected future fundamentals converges.) Then we can write our solution as
\[
p_t = \frac{1}{1 + \lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^i \mathcal{E}_t \tilde{m}_{t+i}
\]  

(19)

3.1.1 Bubbles

Noting that \( \frac{\lambda}{1 + \lambda} < 1 \), we assumed above that \( \lim_{n \to \infty} \frac{\lambda}{(1 + \lambda)^n} \mathcal{E}_t p_{t+n+1} = 0 \). However, we can imagine price level “bubbles” that would generate a non-zero limit for this term. In this section we offer a simple illustration of this possibility.

Let \( p_f^t \) be the present value solution. That is
\[
p_f^t = \frac{1}{1 + \lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^i \mathcal{E}_t \tilde{m}_{t+i}
\]  

(20)

Next define a bubble \( b_t \) as the following explosive first-order autoregressive process:
\[
b_t = \frac{1 + \lambda}{\lambda} b_{t-1} + \eta_t
\]  

(21)

Here \( \eta_t \) is white noise (i.e., mean zero and constant variance). So
\[
\mathcal{E}_t b_{t+1} = \frac{1 + \lambda}{\lambda} b_t
\]  

(22)

We propose that \( p_f^t + b_t \), which we will call the bubble solution, is a solution to our rational expectations model. We check this by seeing if it satisfies (13). That is, does our proposed solution satisfy (13)?
\[
p_f^t + b_t \equiv \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t \left( p_f^{t+1} + b_{t+1} \right) 
\]  

(23)

\[
b_t \equiv \frac{\lambda}{1 + \lambda} \mathcal{E}_t b_{t+1} 
\]  

(24)

\[
b_t = b_t
\]  

(25)

The last equality follows from (22). So we see that the bubble solution is also a solution to our model.

3.2 Another Representation of the Algebra

Use of the “forward shift operator” offers a simplified representation of the algebra. To reduce notational clutter, let \( \mu = \lambda/(1 + \lambda) \) and rewrite our characterization of the price level in terms of fundamentals and expectations as
\[
p_t = \mathcal{E}_t \{(1 - \mu)\tilde{m}_t + \mu p_{t+1}\}
\]  

(26)

That is:
\[
p_t = (1 - \mu)\tilde{m}_t + \mu \mathcal{E}_t p_{t+1}
\]  

(27)

Since the expectation at time \( t \) of the current price level is just the current price level, and similarly for the current money supply, we can rewrite this as
\[
\mathcal{E}_t p_t = (1 - \mu)\tilde{m}_t + \mu \mathcal{E}_t p_{t+1}
\]  

(28)

Now consider the forward shift operator, \( F \), defined as \( x_{t+n} = F^n x_t \), and use it to rewrite (28) as
\[
(1 - \mu F) \mathcal{E}_t p_t = (1 - \mu)\tilde{m}_t
\]  

(29)

Now if we could just multiply both sides by the inverse of \( (1 - \mu F) \), we would have a solution for the price level.
More accurately, we would have
\[ E_t p_t = (1 - \mu F)^{-1} (1 - \mu) E_t \tilde{m}_t + \eta \mu^{-t} \]

The term \( \eta \mu^{-t} \) is permitted in the solution, since
\[ (1 - \mu F) \eta \mu^{-t} = 0 \]
However we will ignore such “bubbles” in the solution.

\[ E_t p_t = (1 - \mu F)^{-1} (1 - \mu) E_t \tilde{m}_t \tag{30} \]

It turns out that we can do this. Note that \((1 - \mu F)(1 + \mu F + \mu^2 F^2 + \cdots) = 1\). Therefore, define the inverse to be
\[ (1 - \mu F)^{-1} = 1 + \mu F + \mu^2 F^2 + \cdots = \sum_{i=0}^{\infty} \mu^i F^i \tag{31} \]
So we can write our reduced form for the price level as
\[ E_t p_t = (1 - \mu) \sum_{i=0}^{\infty} \mu^i E_t \tilde{m}_{t+i} \tag{32} \]
Of course \( E_t p_t = p_t \), so we have found our solution for the price level.

## 4 An Observable Solution

We have seen that the price level solution is a weighted sum of expected future fundamentals. Let us turn to the question of how to use this solution in empirical work. How can we handle the expected future fundamentals in the exchange rate solution? Until we can relate these expected future values to something we can observe and measure, our price level solution under rational expectations cannot be turned into a useful empirical model. For example, we cannot offer any simple relationship between the price level and the money supply until we know how the expected future behavior of the money supply is related to its current and past behavior. Many economists approach this problem by characterizing the way the fundamentals evolve over time. Such a characterization is called a data generating process (DGP) for the fundamentals. This information can then be used in forming expectations about the future.

A common assumption is that the price level fundamentals follow a random walk (with drift):
\[ \tilde{m}_t = \tilde{m}_{t-1} + u_t \tag{33} \]
Here \( u_t \) is the unanticipated change in the fundamentals, which averages zero (and is not serially correlated). This just means that the fundamentals are just as likely to rise as to fall each period. In this case our “best guess” of the future fundamentals is their current value. That is
\[ E_t \tilde{m}_{t+i} = \tilde{m}_t \tag{34} \]
If the fundamentals are just as likely to rise as to fall, so is the price level. That is, our best guess of the future price level is the current price level. Recalling (6), we can conclude that
\[ p_t = \tilde{m}_t \tag{35} \]
(This can also be seen algebraically by substituting (34) into (11).) Recalling the definition of the price level fundamentals, we can then gather data and test (35) empirically.
\[ p_t = m - \phi y \tag{36} \]
Note how similar this is to the crude “Classical” model: only the interest rate effect has disappeared.

Obviously the fundamentals cannot always be characterized as following a random walk. For example, if one country consistently has high inflation and another country consistently has low inflation, the random walk characterization of the price level fundamentals for these two countries will be a poor one. As a result, for many price levels we need to characterize the price level fundamentals by a more complicated data generating process. For example, we may assume that the fundamentals follow a simple autoregressive process.

A more typical way of closing a rational expectations model is much more general: assume that the \( n \) exogenous variables follow a \( k \)th-order vector autoregressive process (VAR) after differencing. This is a bit more complicated, so details will be relegated to appendix A. Briefly, let \( X_t \) be the \( n \)-vector of exogenous variables at time \( t \). For example, we might treat money supplies and incomes as the only components of our fundamentals. Let \( X_t^\top = (m, y) \), so we can rewrite our fundamentals as \( \tilde{m}_t = a^\top X_t \) where \( a^\top = (1, -\phi) \). The VAR process simply relates \( X_t \) to its past values.

\[
\Delta X_t = \sum_{j=1}^{k} B_j \Delta X_{t-j} + v_t \tag{37}
\]

The \( B_j \)'s are the matrices of coefficients on the lagged exogenous variables and \( v_t \) is a vector of errors. If we create a vector \( Z_t \) containing the current and lagged exogenous variables, then as shown in appendix A, our solution for the price level is a linear function of the current and lagged exogenous variables.

\[
p_t = a^\top X_t + a^\top GC \Delta Z_t \tag{38}
\]

where \( GC \) is a matrix defined in appendix A. This can be estimated simultaneously with (67).

Even this considerably complicates the problem of representing and summing up the expected future fundamentals. One useful alternative method for dealing with more general DGPs is the method of undetermined coefficients, which is briefly treated below.

5 The Data Generating Process

Before starting on the algebra, let us get our bearings. Remember, we have a model of the price level under rational expectations

\[
p = \frac{1}{1+\lambda} (\tilde{m} + \lambda \mathbb{E}_t p) \tag{39}
\]

that provides the basic structure of price level determination. We found that we could solve this model for the price level as

\[
p_t = \frac{1}{1+\lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^i \mathbb{E}_t \tilde{m}_{t+i} \tag{40}
\]

where \( \tilde{m} \) represents the price level fundamentals (e.g., money supplies and the determinants of real money demand). Now the problem with this kind of formulation is that it still involves unobserved expectations. While it is informative at the theoretical level, it is not very useful empirically. We would like to have an observable solution for the price level, a solution in terms of variables we can observe and measure. To move from a solution like (40) to an observable solution, we represent the fundamentals by a data generating process (DGP).

For example, we initially treated \( \tilde{m} \) as following a random walk, which yielded a simple observable solution for the price level. But while this may be a good approximation for some pairs of countries, for others it will be terrible. (For example, one country might have persistent high inflation when the other does not.) So consider a more general DGP: the following simple “autoregressive” representation of the fundamentals.

\[
\tilde{m}_t = \mu_0 + \mu_1 \tilde{m}_{t-1} + u_t \tag{41}
\]

We will show that given this AR(1) DGP, the price level solution is

\[
p_t = \frac{\lambda \mu_0}{1+\lambda - \lambda \mu_1} + \frac{1}{1+\lambda - \lambda \mu_1} \tilde{m}_t \tag{42}
\]
5.1 Anticipating Future Fundamentals

Now if we know the AR(1) DGP governs the evolution of the fundamentals over time, then in the future we will have

\[
\begin{align*}
\tilde{m}_{t+1} &= \mu_0 + \mu_1 \tilde{m}_t + u_{t+1} \\
\tilde{m}_{t+2} &= \mu_0 + \mu_1 \tilde{m}_{t+1} + u_{t+2} \\
\tilde{m}_{t+3} &= \mu_0 + \mu_1 \tilde{m}_{t+2} + u_{t+3}
\end{align*}
\]

(43)

e etc.

Forming our expectations in light of this knowledge tells us

\[
\begin{align*}
\mathcal{E}_t \tilde{m}_{t+1} &= \mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_t + \mathcal{E}_t u_{t+1} \\
&= \mu_0 + \mu_1 \tilde{m}_t
\end{align*}
\]

(44)

\[
\begin{align*}
\mathcal{E}_t \tilde{m}_{t+2} &= \mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_{t+1} + \mathcal{E}_t u_{t+2} \\
&= \mu_0 + \mu_1 (\mu_0 + \mu_1 \tilde{m}_t) \\
&= \mu_0 + \mu_1 \mu_0 + \mu_1^2 \tilde{m}_t
\end{align*}
\]

(45)

\[
\begin{align*}
\mathcal{E}_t \tilde{m}_{t+3} &= \mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_{t+2} + \mathcal{E}_t u_{t+3} \\
&= \mu_0 + \mu_1 (\mu_0 + \mu_1 \mu_0 + \mu_1^2 \tilde{m}_t) \\
&= \mu_0 + \mu_1 \mu_0 + \mu_1^2 \mu_0 + \mu_1^3 \tilde{m}_t
\end{align*}
\]

(46)

\[
\begin{align*}
\vdots
\end{align*}
\]

(47)

We quickly see a pattern emerging, which can be summarized by (48).

\[
\begin{align*}
\mathcal{E}_t \tilde{m}_{t+i} &= \mu_0 \sum_{j=0}^{i-1} \mu_1^j + \mu_1^i \tilde{m}_t \\
i &\geq 1
\end{align*}
\]

(48)

This gives us an important insight. All of our expectations of future fundamentals can be stated in terms of the current fundamentals.

5.2 Finding An Observable Reduced Form: The Algebra

In principle we could substitute our solutions (48) for expected future fundamentals into equation (40) and actually perform the summation. This can be very messy, however, so often it is convenient to rely instead on the method of undetermined coefficients. We will do it both ways.

5.2.1 Direct Summation

Let us continue to use the convenience notation \( \mu = \lambda/(1 + \lambda) \).

\[
\begin{align*}
p_t &= (1 - \mu) \sum_{i=0}^{\infty} \mu^i \mathcal{E}_t \tilde{m}_{t+i} \\
&= (1 - \mu) \sum_{i=1}^{\infty} \mu^i (\mathcal{E}_t \tilde{m}_{t+i}) + (1 - \mu) \tilde{m}_t \\
&= (1 - \mu) \sum_{i=1}^{\infty} \mu^i \left( \mu_0 \sum_{j=0}^{i-1} \mu_1^j + \mu_1^i \tilde{m}_t \right) + (1 - \mu) \tilde{m}_t \\
&= (1 - \mu) \sum_{i=1}^{\infty} \mu^i \sum_{j=0}^{i-1} \mu_1^j + (1 - \mu) \sum_{i=0}^{\infty} \mu^i \mu_1^i \tilde{m}_t
\end{align*}
\]

(49)
So we have

\[ p_t = \phi_0 + \phi_1 \tilde{m}_t \]  

(50)

where \( \phi_0 \) and \( \phi_1 \) are the constant coefficients that are the infinite sums of structural form parameters in (49). It turns out that we can give much simpler representations of these coefficients.

Noting that

\[ \sum_{i=0}^{\infty} \mu^i \mu_1^i = \frac{1}{1 - \mu \mu_1} \]  

(51)

we have

\[ \phi_1 = \frac{1 - \mu}{1 - \mu \mu_1} \]  

(52)

Finding \( \phi_0 \) is a little more work. Let us begin with the inner summation.

\[ \sum_{j=0}^{i-1} \mu_1^j = \frac{1}{1 - \mu \mu_1} (1 - \mu_1^i) \]  

(53)

Then we can write

\[ \phi_0 = \frac{1 - \mu}{1 - \mu_1} \mu_0 \sum_{i=1}^{\infty} \mu^i (1 - \mu_1^i) \]  

(54)

Now note that

\[ \sum_{i=1}^{\infty} \mu^i (1 - \mu_1^i) = \sum_{i=1}^{\infty} \mu^i - \sum_{i=1}^{\infty} \mu^i \mu_1^i \]

\[ = \frac{\mu}{1 - \mu_1} - \frac{\mu \mu_1}{1 - \mu \mu_1} \]  

(55)

\[ = \frac{\mu (1 - \mu_1)}{(1 - \mu)(1 - \mu \mu_1)} \]

So we get

\[ \phi_0 = \frac{1 - \mu}{1 - \mu_1} \mu_0 \frac{\mu (1 - \mu_1)}{(1 - \mu)(1 - \mu \mu_1)} \]

\[ = \frac{\mu \mu_0}{1 - \mu \mu_1} \]  

(56)

Finally, since we have solved for \( \phi_0 \) and \( \phi_1 \), we can write

\[ p_t = \frac{\mu \mu_0}{1 - \mu \mu_1} + \frac{1 - \mu}{1 - \mu_1} \tilde{m}_t \]  

(57)

5.2.2 The Method of Undetermined Coefficients

Our second approach to finding an observable reduced form will be the method of undetermined coefficients. (Naturally, it will yield the same solution.) The method of undetermined coefficients requires that we guess the general form of our solution. However the guess is not arbitrary. Instead it relies on two pieces of information. First, we saw in equation (40) that we can express the current price level in terms of expected future fundamentals. Second, we found that all expected future fundamentals can be expressed in terms of current fundamentals. These two pieces of information suggest that the price level can be expressed in terms of the current fundamentals. This is captured by the following expression.

\[ p_t = \phi_0 + \phi_1 \tilde{m}_t \]  

(58)

Unfortunately, this expression involves two undetermined coefficients, \( \phi_0 \) and \( \phi_1 \). We would like to know how these are related to the underlying structural parameters. We can get information about this by turning once again to equation (9). First note that our guessed solution implies

\[ p_{t+1} = \phi_0 + \phi_1 \tilde{m}_{t+1} \]  

(59)
So
\[
\mathcal{E}_t p_{t+1} = \phi_0 + \phi_1 \mathcal{E}_t \tilde{m}_{t+1}
\]
\[
= \phi_0 + \phi_1 (\mu_0 + \mu_1 \tilde{m}_t)
\]
(60)

So we can substitute our expressions for \( p \) and \( p^e \) in the undetermined coefficients into equation (9) to get
\[
p_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t p_{t+1}
\]
(61)
\[
\phi_0 + \phi_1 \tilde{m}_t \equiv \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} [\phi_0 + \phi_1 (\mu_0 + \mu_1 \tilde{m}_t)]
\]
(62)
\[
= \frac{\lambda}{1 + \lambda} (\phi_0 + \phi_1 \mu_0) + \frac{1 + \lambda \phi_1 \mu_1}{1 + \lambda} \tilde{m}_t
\]

Now here is the tricky part. Equation (62) must hold for any level of the fundamentals \( \tilde{m}_t \). That is what allows us to determine the \( \phi \)s: the slope and intercept must be the same for the function of \( \tilde{m}_t \) that we find on each side of the equality (62).
\[
\phi_0 \equiv \frac{\lambda}{1 + \lambda} (\phi_0 + \phi_1 \mu_0)
\]
(63)
\[
\phi_1 \equiv \frac{1 + \lambda \phi_1 \mu_1}{1 + \lambda}
\]

These are two equations in the two unknowns \( \phi_0 \) and \( \phi_1 \). The second equation can be solved for
\[
\phi_1 = \frac{1}{1 + \lambda - \lambda \mu_1}
\]
(64)
so the first gives us
\[
\phi_0 = \frac{\lambda \mu_0 \phi_1}{1 + \lambda - \lambda \mu_1}
\]
(65)
\[
= \frac{\lambda \mu_0}{1 + \lambda - \lambda \mu_1}
\]

So we can now replace the undetermined coefficients in our guess (58) to get
\[
p_t = \frac{\lambda \mu_0}{1 + \lambda - \lambda \mu_1} + \frac{1}{1 + \lambda - \lambda \mu_1} \tilde{m}_t
\]
(66)

In empirical work, we can then estimate (66) and our AR(1) DGP simultaneously.

Equation (66) has an important message for empirical research. There is no simple relationship between price levels and money supplies. Note that the relationship between the price level and the money supply depends on all the parameters of the data generating process. For example, if \( \mu_1 < 1 \) so that money supply increases tend to be reversed over time, then the price level will rise less than in proportion to a money supply increase.

In summary, the method of undetermined coefficients proceeds as follows:

1. Begin with a model and a data generating process (DGP) for the exogenous variables.

2. Based on the DGP, make an educated guess about the functional form of the solution, which will generally involve the exogenous variables, shocks to the system, and possibly lagged endogenous variables. That is, guess an “observable reduced form” for the system, where your guess involves “undetermined coefficients” that you want to express in terms of structural coefficients.

3. Find the expectations implied by your proposed solution form.

4. Plug these expectations into the model.

5. Use the implied identities in the coefficients to solve for the undetermined coefficients in terms of the structural coefficients (here, the \( \mu \)s and \( \lambda \)).
6 Concluding Comments

A key lesson of this chapter is that there is no simple relationship between changes in the money supply and changes in the price level. The effect of a change in the current money supply on the current price level depends crucially on its effect on the expected future money supply. So beliefs about the monetary policy “reaction function” will be an important determinant of the contemporary link between prices and money supplies.

In the Classical model, nominal interest rates are determined by expected inflation. If individuals accurately link expected inflation to their anticipations of monetary policy, we can make more detailed predictions about the behavior of prices over time. This observation carries over immediately to the “Classical” to price level determination, since in the monetary approach PPP ensures that any predictions we make about the behavior of prices are also predictions about the behavior of the price level.

A The Data Generating Process: Detailed Analysis

A typical way of closing a rational expectations model is by assuming that the $n$ exogenous variables follow a $k^{th}$-order vector autoregressive process (VAR) after differencing. The following treatment provides some general tools for implementing this closure.\(^1\) Let $X_t$ be the $n$-vector of exogenous variables at time $t$.

$$\Delta X_t = \sum_{j=1}^{k} B_j \Delta X_{t-j} + v_t$$  \hspace{1cm} (67)

The $B_j$s are the $(n \times n)$ matrices $\{b_{ij\cdot},j\}$, and $v_t$ is an $n$-vector of serially uncorrelated errors. Equation (67) can be rewritten as a first-order VAR as follows.

$$\Delta Z_t = A \Delta Z_{t-1} + \delta_t$$  \hspace{1cm} (68)

where,

$$Z_t^\top = (X_{1,t}, \ldots, X_{1,t-k+1}, X_{2,t}, \ldots, X_{2,t-k+1}, \ldots, X_{n,t}, \ldots, X_{n,t-k+1})$$

$$\delta_t = (v_{1,t}, 0, \ldots, 0, v_{2,t}, 0, \ldots, 0, \ldots, v_{n,t}, 0, \ldots, 0)$$

and, $A$ is the $nk \times nk$ matrix whose $(i-1)k+1^{th}$ row is the column vectorization of the matrix formed by vertically concatenating the $i^{th}$ row of all the $B_j$s, with the rest of the elements zero except for the $k-1$

---

\(^1\) Much of the development below follows Driskill.mark sheffrin (1992 IER) fairly closely.
identity matrices beginning below each $b_{i,1}$.

$$A = \begin{bmatrix}
  b_{1,1} & \ldots & b_{1,k} & b_{12,1} & \ldots & b_{12,k} & \ldots & b_{1n,1} & \ldots & b_{1n,k} \\
  1 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  b_{21,1} & \ldots & b_{21,k} & b_{22,1} & \ldots & b_{22,k} & \ldots & b_{2n,1} & \ldots & b_{2n,k} \\
  0 & \ldots & 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
  0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  b_{n1,1} & \ldots & b_{n1,k} & b_{n2,1} & \ldots & b_{n2,k} & \ldots & b_{nn,1} & \ldots & b_{nn,k} \\
  0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
  0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 1 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots \\
  0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 1 & 0 
\end{bmatrix}$$

We then calculate the $j$-step ahead linear least squares predictor $\mathcal{E}_t Z_{t+j}$ as follows. We can write,

$$\begin{align*}
\mathcal{E}_t Z_{t+j} &= \mathcal{E}_t [Z_{t+j} - (Z_{t+j-1} + \ldots + Z_t) + (Z_{t+j-1} + \ldots + Z_t)] \\
&= \mathcal{E}_t [\Delta Z_{t+j} + \Delta Z_{t+j-1} + \ldots + \Delta Z_{t+1} + Z_t] \\
&= \sum_{i=0}^{j-1} \mathcal{E}_t \Delta Z_{t+j-i} + Z_t
\end{align*}
$$

(69)

Now from the data generating process (68), we know

$$\begin{align*}
\mathcal{E}_t \Delta Z_{t+j} &= A \mathcal{E}_t \Delta Z_{t-1+j} \\
&= A^2 \mathcal{E}_t \Delta Z_{t-2+j} \\
&= \ldots \\
&= A^j \Delta Z_t
\end{align*}
$$

(70)

Combining (69) with equation (70) we get;

$$\begin{align*}
\mathcal{E}_t Z_{t+j} &= Z_t + \sum_{i=0}^{j-1} A^{j-i} Z_t \\
&= Z_t + \sum_{i=1}^{j} A^i \Delta Z_t
\end{align*}
$$

(71)

This is the analogue to the scalar result. Recall

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}
$$

The sum from 1 to $j$ is $a$ times the sum from 0 to $j - 1$.

$$\sum_{i=1}^{j} A^i = A(I - A^j)(I - A)^{-1}
$$

$$\therefore \mathcal{E}_t Z_{t+j} = Z_t + A(I - A^j)(I - A)^{-1} \Delta Z_t
$$

(72)
Equation (72) is the $j$-step ahead linear least squares predictor of $Z_t$.

Let $G$ be a $(n \times nk)$ matrix of zeros, except for the elements $g_{11}, g_{2,k+1}, g_{3,2k+1}$, etc. which are equal to one, then $X_t = GZ_t$. Hence, using equation (72) we get the $j$-step ahead linear least squares predictor of $X_t$ as follows.

\[ E_t(X_{t+j}) = GZ_t + GA(I_{nk} - A^j)(I_{nk} - A)^{-1}\Delta Z_t \quad (73) \]

One more preliminary. It will be useful to note that

\[ A \sum_{j=0}^{\infty} \mu^j (I - A^j)(I - A)^{-1} = A \left( \sum_{j=0}^{\infty} \mu^j I - \sum_{j=0}^{\infty} \mu^j A^j \right) (I - A)^{-1} \]

\[ = \left[ \frac{1}{1-\mu} I - (I - \mu A)^{-1} \right] A(I - A)^{-1} \]

\[ = \frac{1}{1-\mu} (I - \mu A)^{-1}[(I - \mu A) - (1 - \mu)I]A(I - A)^{-1} \]

\[ = \frac{1}{1 - \mu} (I - \mu A)^{-1}[\mu(I - A)]A(I - A)^{-1} \]

\[ = \frac{\mu}{1 - \mu} A(I - \mu A)^{-1} \]

Letting

\[ C = \mu A(I - \mu A)^{-1} \quad (75) \]

we can express this result as

\[ A \sum_{j=0}^{\infty} \mu^j (I - A^j)(I - A)^{-1} = \frac{1}{1-\mu} C \quad (76) \]

**The Observable Reduced Form**

Recall that the semi-reduced form price level equation was given by

\[ p_t = (1 - \mu) \sum_{i=0}^{\infty} \mu^i E_t \tilde{m}_{t+i} \]

where $\tilde{m}$ includes $m - \phi y$ and $\mu = \lambda/(1 + \lambda)$. Let us now represent money supplies and incomes, our two exogenous variables, to follow a $k$-th order VAR, as discussed above. In the notation of the previous section, we have our price level solution is

\[ p_t = (1 - \mu)a' \sum_{i=0}^{\infty} \mu^i E_t X_{t+i} \]

where

\[ a = \begin{bmatrix} 1 \\ -\phi \end{bmatrix} \quad \text{and} \quad X_t = \begin{bmatrix} m \\ y \end{bmatrix} \]

From our results (73) and (76), we have

\[ \sum_{i=0}^{\infty} \mu^i E_t X_{t+i} = \sum_{i=0}^{\infty} \mu^i [GZ_t + GA(I_{nk} - A^i)(I_{nk} - A)^{-1}\Delta Z_t] \]

\[ = \frac{1}{1 - \mu} G(Z_t + C\Delta Z_t) \]

We therefore have our solution for the price level:

\[ p_t = a' G(Z_t + C\Delta Z_t) \]

\[ = a' X_t + a' G C \Delta Z_t \quad (77) \]

This can be estimated simultaneously with (67). Note that while the equation for the price level is linear in the exogenous variables, it is non-linear in the structural parameters (because of $C$). So if you wish to estimate the structural parameters, you will need to use a method that can account for this non-linearity.
**B Partial Adjustment of Money Demand**

In this handout, we have continued to treat the current value of real money demand as a simple function of current income and interest rates. However, and as Bilson (1978, Frenkel and Johnson) and Woo (1985 JIE) emphasize, applied work on money demand generally uses this functional form only for “long-run” money demand. Allowing for a partial adjustment characterization of money demand yields the equation

\[ p_t = m_t - \phi y_t + \lambda r_t + \lambda (p^e_t - p_t) - \alpha m_{t-1} + \alpha p_{t-1} \]

When combined with the Fisher effect, we then have a new price level equation

\[ -\lambda p^e_t + (1 + \lambda) p_t - \alpha p_{t-1} = m_t - \phi y_t + \lambda r_t - \alpha m_{t-1} \]

We proceed with the solution under the assumption of rational expectations. Let \( X_t^T = (m_t, y_t) \), \( a^T = (-1, \phi) \), and \( b^T = (\alpha, 0) \). (We will ignore the non-observable elements of \( \tilde{m} \) to simplify the algebra.) Then letting \( \tilde{m}_t = a^T X_t + b^T X_{t-1} \) we can write

\[ \lambda E_t p_t + 1 - (1 + \lambda) E_t p_t + \alpha E_t p_t - 1 = -m_t + \phi y_t - \lambda r_t + \alpha (m_{t-1}) \]

\[ = a^T X_t + b^T X_{t-1} \]

Now transform this by taking expectations at time \( t \).

\[ [\lambda F - (1 + \lambda) + \alpha F^{-1}] E_t p_t = E_t \tilde{m}_t \] (78)

With a slight abuse of notation, we will say the characteristic equation is

\[ \lambda F^2 - (1 + \lambda) F + \alpha = 0 \]

with solutions

\[ F_1, F_2 = \frac{1 + \lambda \pm \sqrt{(1 + \lambda)^2 - 4\lambda \alpha}}{2\lambda} \]

Thus we can rewrite (78) as

\[ (F - F_1)(F - F_2)\lambda F^{-1} E_t p_t = E_t \tilde{m}_t \]

which has the general solution

\[ \lambda F^{-1} E_t p_t = (F - F_1)^{-1} (F - F_2)^{-1} E_t \tilde{m}_t + c_1 F_1^t + c_2 F_2^t \] (79)

from the general solution, it is clear that the price level will tend to move explosively away from the fundamentals if either root is greater than unity in absolute value. Assuming that the parameters have the expected signs and magnitudes, \( \lambda > 0 \) and \( \alpha \in (0, 1) \), there are two positive real roots.\(^2\) In addition, the smaller root is less than unity while the larger root is greater than unity. Probably the easiest way to see this is to note that \( dF_1/d\alpha > 0 \) and \( dF_2/d\alpha < 0 \), and then consider the values of \( F_1 \) and \( F_2 \) at the extreme values of \( \alpha \).

<table>
<thead>
<tr>
<th>Characteristic Roots: Relative Magnitudes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta \geq 1 )</td>
</tr>
<tr>
<td>( F_1 )</td>
</tr>
<tr>
<td>( \alpha = 0 )</td>
</tr>
<tr>
<td>( \alpha = 1 )</td>
</tr>
</tbody>
</table>

\(^2\)With \( \lambda, \alpha > 0 \) the requirement that \((1 + \lambda)^2 - 4\lambda \alpha > 0 \) is satisfied as long as \( \lambda + 1/\lambda > 4\alpha - 2 \). Given the nature of the partial adjustment mechanism (i.e., \( \alpha < 1 \)), this is necessarily satisfied.
Thus we have a situation of saddle-path stability. Here is how we will deal with the instability deriving from $F_2$: we will set $c_2 = 0$. This is known as a “transversality condition”; it assures us that the price level will approach its fundamentals in the long run. Putting it another way, it is ruling out explosive price level bubbles. The transversality condition is serving another important role for us in our solution: without it, we do not have enough information to determine a unique price level. That is because we are working with a second order difference equation, but we only have a single initial condition ($p_{t-1}$). We can highlight the role of this initial condition by multiplying (79) by $(F - F_1)$.

$$(F - F_1)\lambda F^{-1} \mathcal{E}_t p_t = (F - F_2)^{-1} \mathcal{E}_t \tilde{m}_t$$

Noting $F_1 F_2 = \alpha/\lambda$, we can write this as

$$(1 - \frac{1}{F_1} F) \alpha F^{-1} \mathcal{E}_t p_t = (1 - \frac{1}{F_2} F)^{-1} \mathcal{E}_t \tilde{m}_t$$

or, using summation notation,

$$\mathcal{E}_t p_{t-1} - \frac{1}{F_1} \mathcal{E}_t p_t = \frac{1}{\alpha} \sum_{i=0}^{\infty} \left( \frac{1}{F_2} \right)^i \mathcal{E}_t \tilde{m}_{t+i}$$

$$\mathcal{E}_t p_t = F_1 \mathcal{E}_t p_{t-1} - \frac{F_1}{\alpha} \sum_{i=0}^{\infty} \left( \frac{1}{F_2} \right)^i \mathcal{E}_t \tilde{m}_{t+i} \quad (80)$$

More Reading

Christiano (1987 IER).