This chapter will introduce you to some basic concepts and solution procedures in the theory of differential equations. It is a "cookbook" in the sense that it lays out a useful solution algorithm that can be followed with only minimal understanding of the underlying principles. However, much of the required background is included in a condensed fashion, and some technical details are included in the endnotes.

Most of our attention will be occupied by elaborations on the following observation: if $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$, then $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$. This is already enough information to solve some differential equations.

Try Exercise #1 now.

If this is your first time through this chapter, please skip immediately to section 4 on the adjoint matrix technique.

1 Fundamental Considerations

Definition: Consider the following first order system of differential equations.[1]

$$\dot{\boldsymbol{x}}(t) = f[\boldsymbol{x}(t), t] \tag{1}$$

Here $\dot{\boldsymbol{x}} \equiv d\boldsymbol{x}/dt$ and $f(\cdot)$ is a vector of n real valued functions. (We will think of t as representing time.)

Note: an n-th order differential equation,

$$y^{(n)} = F[y(t), \dot{y}(t), \ldots, y^{(n-1)}, t]$$

can always be written as a system of n first order differential equations by defining new variables $x_i = y^{(i-1)}$ so that

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \ldots, \dot{x}_{n-1} = x_n, \dot{x}_n = F[x_1, x_2, \ldots, x_n, t]$$

This is a system of n first order differential equations equivalent to the original n-th order equation. So we often discuss only first order systems.

Definition: a *solution* to a differential equation system is a continuously differentiable *function* of time that satisfies the differential equation system.[2]

[1] Here $f : X \otimes \boldsymbol{T} \to \Re^n$ where $X \subseteq \Re^n$ and $\boldsymbol{T} = (t^a, t^b) \subseteq \Re$.

[2] More precisely, we have the following definition:
The $\Re^n$ valued function $\phi(t)$ is called a *solution* on some time interval $(t', t'') \subset T$ if

1. $\phi(t) \in X \qquad \forall t \subset (t', t'')$
2. $\phi(t)$ is continuously differentiable
3. $\dot{\phi}(t) = f[\phi(t), t]$ almost everywhere on $t \subset (t', t'')$

Comment: it is common for the dependence on time to be represented more explicity by a forcing function or control function $u(t)$.[3] We will include this convention in our presentation of the fundamental theorem in the theory of differential equations: the Cauchy-Peano Theorem.[4]

Theorem 1 (Cauchy-Peano)

If

$$\dot{x}(t) = f[x(t), u(t), t]$$

then there is a unique definite solution $x^d(t)$ that satisfies the initial condition that $x^d(t^o) = x^o$.[5]

Proof:

See Brock and Malliaris (1989), ch.1.

2 Linear Systems

Definition: Suppose that $\dot{x} = f[x(t), t]$ can be written in matrix form as in equation (2).

$$\dot{x}(t) = A(t)x(t) + u(t) \tag{2}$$

Here $\boldsymbol{A}(t)$ is a (possibly time varying) nxn matrix and $\dot{\boldsymbol{x}}(t)$, $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ are $n \times 1$ vectors. We call (2) a linear non-homogeneous system of differential equations. If $\boldsymbol{u}(t) \equiv 0$, the linear system is homogeneous (of degree one in $\boldsymbol{x}$).

Try Exercise #2 now.

Comment: We are particularly interested in one key fact about linear homogeneous systems: if we can find n linearly independent solutions then we can characterize all possible solutions.

Theorem:

IF $\Phi(t)$ is an $n \times n$ matrix whose columns are linearly independent solutions to the linear homogeneous system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$,

If we can find a solution on the entire time domain T, we call it a global solution.

[3]The forcing function $\boldsymbol{u}(t)$ can be vector valued so that $\boldsymbol{u} : T \to \Re^m$.

[4]The theorem gives a set of *sufficient* conditions for $\dot{\boldsymbol{x}} = f[\boldsymbol{x}(t), \boldsymbol{u}(t), t]$ to have a solution, where $f : X \otimes \Re^m \otimes T \to \Re^n, X \subset \Re^n$ is open and connected, $T = (t^a, t^b) \subset \Re$. The following assumptions then ensure existence and uniqueness.

1. f is continuous on $X \otimes \Re^m \otimes T$ (crucial for existence, may be *slightly* weakened)
2. $\partial f_i / \partial x_j$ exists and is continuous on $X \otimes \Re^m \otimes T$ (useful for uniqueness; a Lipschitz condition suffices)
3. $u(t)$ is piecewise continuous on T
4. $(\boldsymbol{x}^o, t^o) \in X \otimes T$

These assumptions imply that, in some neighborhood of t^o, there is a *locally* unique solution $\boldsymbol{x}^d(t)$ such that $\boldsymbol{x}^d(t^o) = \boldsymbol{x}^o$. For linear systems (which need not have constant coefficients), the result is global.

[5]**Definition:** An **initial condition** is the requirement that the solution pass through specified point $(\boldsymbol{x}^o, t^o) \in X \otimes T$. That is, we require $\boldsymbol{x}^d(t^o) = \boldsymbol{x}^o$. We call $\boldsymbol{x}^o$ the initial value and t^o the intial time.

THEN any solution to the system can be written as $\Phi(t) \cdot c$ for some vector of constants c.[6]

Proof: Since $| \Phi(t) | \neq 0$ we can consider $\Phi^{-1}\psi$ for any solution ψ. Time differentiation yields $\dot{\Phi}^{-1}\psi + \Phi^{-1}\dot{\psi}$. Now since $\Phi\Phi^{-1} = I$ we know

$$\dot{\Phi}\Phi^{-1} + \Phi\dot{\Phi}^{-1} = 0$$

and thus

$$\dot{\Phi}^{-1} = -\Phi^{-1}\dot{\Phi}\Phi^{-1}$$

So

$$\begin{aligned} d(\Phi^{-1}\psi)/dt &= \dot{\Phi}^{-1}\psi + \Phi^{-1}\dot{\psi} \\ &= -\Phi^{-1}\dot{\Phi}\Phi^{-1}\psi + \Phi^{-1}\dot{\psi} \\ &= -\Phi^{-1}(A\Phi)\Phi^{-1}\psi + \Phi^{-1}(A\psi) \\ &= -\Phi^{-1}A\psi + \Phi^{-1}A\psi \\ &= 0 \end{aligned}$$

Thus $\Phi^{-1}\psi = c$, or $\psi = \Phi c$, for some vector of constants c.

Comment: Although there are increasing numbers of exceptions, most economic applications of differential equations sooner or later focus on the analysis of linear systems. In fact, we usually care about problems where $A(t)$ is not time varying. This cookbook therefore focuses on linear systems of constant coefficients.

3 Constant Coefficients

Definition: If $\dot{x} = f[x(t), t]$ can be written in the matrix form

$$\dot{x}(t) = Ax(t) + u(t) \tag{3}$$

[6]If you wish to explore the notion of linear independence of solutions in more detail, see Brock and Malliaris p. 35, Beavis and Dobbs p. 145, or Murata. In short, since any solution $\phi(t)$ is an n-vector we can apply our usual notion of linear independence to a collection of solutions $\Phi(t)$ to discover if at a given time t they generate a set of vectors that are linearly independent. However, we say that the solutions are linearly independent iff no linear combination is null for *all* admissible t. It turns out that uniquenss of the initial value problem ensures that a collection of solutions is linearly dependent either always or never, since the initial value of 0 produces the unique solution $\psi(t) = 0$ which is null at all t. Thus if at any t we know $\Phi(t) \cdot c = 0$, recalling that you have shown as a HW that $\Phi(t) \cdot c$ is a solution, then $\Phi(t) \cdot c = 0$ at all t.

where A is an nxn matrix of constants and $x(t)$ and $u(t)$ are nx1 vectors it is called a linear system with constant coefficients.

The general solution to (3) can be written as[7]

$$x^g(t) = \exp\{At\}k + \exp\{At\}\exp\{-As\}u(s)ds \tag{4}$$

Here k is a vector of arbitrary constants. This solution is general in the sense that it is the functional form that any solution must have. If in addition to equation (3) we are given an initial condition, we can say even more. Then the unique definite solution to the initial value problem is:

$$x^d(t) = \exp\{A(t - t^o)\}x^o + \exp\{At\}\exp\{-As\}u(s)ds \tag{5}$$

This is a global solution because it satisfies (3) at every point in time, and it has a definite value determined by the specified value of x^o.

Try Exercise #3 now.

3.1 Proof of Uniqueness in the Linear Case with Constant Coefficients:

We can easily show the solution to the initial value problem is unique for any linear differential equation system of constant coefficients *if* we know that the linear *homogeneous* system has a unique solution satisfying a given initial condition.

Theorem: The initial value problem,

Solve $\dot{x}(t) = Ax(t)$ such that $x(t^o) = x^o$

has the unique definite solution:

$$x(t) = \exp\{A(t - t^o)\}x^o$$

Proof: See footnote.[8]

[7] Similar results obtain for the systems with non-constant coefficients, where $\exp\{At\}$ is replaced by a matrix of n linearly independent solutions. In systems with constant coefficients $\exp\{At\}$ is such a matrix, but with time varying coefficents we cannot just use $\exp\{\int A(s)ds\}$ except in special cases. See Brock and Malliaris ch.2.

[8] Proof: Differentiation proves that $\exp\{At\}\boldsymbol{x}^o$ is a solution. To prove this solution is unique, consider an arbitrary solution $\psi(t)$. Define $\boldsymbol{y}(t)$ by

$$\boldsymbol{y}(t) \equiv \exp\{-At\}\psi(t) \tag{***}$$

Then

$$\begin{aligned}\dot{\boldsymbol{y}}(t) &= -A\exp\{-At\}\psi(t) + \exp\{-At\}\dot{\psi}(t)\\ &= \exp\{-At\}(-A)\psi(t) + \exp\{-At\}A\psi(t)\\ &= \exp\{-At\}(-A + A)\psi(t)\\ &= 0\end{aligned}$$

So $\boldsymbol{y}(t)$ is constant and, since $y(0) = \psi(0)$, from the assumption that $\psi(t)$ solves the initial value problem we know that

Comment: By convention we usually set $t^o = 0$ and therefore write the unique solution to the initial value problem in the homogeneous case as:

$$x(t) = \exp\{At\}x^o$$

We will use this uniqueness result to prove the uniqueness of the solution $x^d(t)$ to the initial value problem for a non-homogeneous system. Define $x^p(t)$ to be a second solution to (3), in additon to $x^d(t)$ as defined in (5) above, of the complete (non- homogenous) system with $x^p(t^o) = y^o$. Then (3) implies that $x^p(t) - x^d(t)$ solves a new *homogeneous* system: $\dot{x}(t) = Ax$ with $x(t^o) = y^o - x^o$. Therefore, we can use our uniqueness result for linear homogeneous first order systems to write $x^d(t) - x^p(t) = \exp\{A(t-t^o)\}(x^o - y^o) = 0$ if $y^o = x^o$ [i.e., $x^p(t) = x^d(t)$]

This completes the proof that our solution (5) is in fact the unique solution to (3) that satisfies the initial condition.

The uniqueness proof also provides a basis for an alternative solution technique: if we can find *any* particular solution to (3), $x^p(t)$, then the definite solution, $x^d(t)$, can be written as that particular solution plus a general solution to the homogeneous part that we call the complementary solution $x^c(t)$. In other words,

$$x^d(t) - x^p(t) = \exp\{At\}[x^d(0) - x^p(0)]$$
$$\therefore x^d(t) = x^p(t) + \exp\{At\}\eta$$

for some vector of constants, η. This suggests four steps to a solution.

1. find a general solution $x^c(t)$ to the homogeneous part (by general I mean a representation of all possible solutions)

2. find a particular solution $x^p(t)$ to (3) (e.g., by guessing it has the same functional form as the forcing function)

3. find a general solution to (3): $x^g(t) = x^p(t) + x^c(t)$.

4. find the unique definite solution $x^d(t)$ by solving for the constants, η, using the initial condition.

Try Exercise #4 now.

$\boldsymbol{y}(t) = \boldsymbol{x}^o$. Therefore from (***),

$$\psi(t) = \exp\{At\}\boldsymbol{x}^o$$

Q.E.D.

Of course, in the multivariate case there remains the problem of calculating the matrix exponential in (4), which can be difficult.[9] In practice, we often find ad hoc solution proceedures most convenient. The adjoint matrix technique is particularly useful.[10]

4 The Adjoint Matrix Technique

Consider the first-order linear differential equation

$$\dot{x} = Ax \tag{8}$$

where A is square matrix of real constants. Suppose we can find a scalar λ with associated vector κ such that $A\kappa = \lambda\kappa$.[11] Then $x = \exp\{\lambda t\}\kappa$ is a solution, since it implies $\dot{x} = \exp\{\lambda t\}\lambda\kappa = \exp\{\lambda t\}A\kappa = Ax$. The adjoint matrix technique is just an elaboration on this observation. This technique allows us to determine the general solution to the homogeneous part of a system of linear differential equations.

Consider a homogeneous system of linear differential equations (not necessarily first-order) with constant coefficients:

$$P(D)x(t) = 0 \tag{9}$$

where $P(D)$ is a matrix of polynomials in D, the differential operator (i.e. $Dx(t) = dx(t)/dt = \dot{x}(t)$, $D^2x(t) = d^2x(t)/dt^2$, etc). We want to find the general solution for this sytem. The following discussion motivates the ultimate result in some detail, so you may find it useful on first reading to skip immediately to equation (14).

Let λ be a constant and let v be any nonzero $n \times n$ matrix independent of t. Note that

$$P(D)e^{\lambda t}v \equiv e^{\lambda t}P(\lambda)v, \tag{10}$$

[9] However if A has n distinct characteristic roots, we are in luck. Define

$$\chi = [\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n] \tag{6}$$

where $A\boldsymbol{x}^i = \lambda_i \boldsymbol{x}^i$, i.e., the x^i and λ_i are the characteristic vectors and the characteristic roots of A. Then

$$\chi^{-1}A\chi = [\lambda_i \delta_{ij}] \equiv D \tag{7}$$

where $\delta_{ij} =_0^1 \;{}^{i=j}_{i\neq j}$ is diagonal. Now define $\boldsymbol{z}(t) = \chi^{-1}\boldsymbol{x}(t)$ and solve the transformed system:

$$\dot{\boldsymbol{z}}(t) = D\boldsymbol{z}(t) + \chi^{-1}\boldsymbol{u}(t)$$

with $z(t^o) = \chi^{-1}\boldsymbol{x}^o$; as above. This is easy since D is diagonal.

[10] See Murata, Yasuo, Mathematics for the Stability and Optimization of Economic Systems (New York: Academic Press, 1977) for additional details.

[11] If $A\kappa = \lambda\kappa$, we call λ an eigenvalue of A and κ is an associated eigenvector. The eigenvalues and eigenvectors characterize many of the important properties of the matrix A. We find the set of eigenvalues by noticing $(A - \lambda I)\kappa = 0$ only if $\det(A - \lambda I) = 0$, which is called the characteristic equation of A.

Since $P(\lambda)$ is a square matrix of constants, so is its adjoint $P^{\dagger}(\lambda)$. Recall, $P(\lambda)P^{\dagger}(\lambda) = |P(\lambda)|\, I$.[12] Then our last result implies

$$\begin{aligned} P(D)e^{\lambda t}P^{\dagger}(\lambda) &= e^{\lambda t}P(\lambda)P^{\dagger}(\lambda) \\ &= e^{\lambda t}|P(\lambda)|I \end{aligned} \tag{11}$$

Now consider the characteristic equation of our differential equation system:

$$\mid P(\lambda) \mid = 0, \tag{12}$$

where $\mid P(\lambda) \mid$ is an n^{th} order polynomial in the variable, λ.

The roots of this equation are called the characteristic roots of the differential equation system. Choosing any root λ_i of the characteristic equation will give us

$$P(D)\exp\{\lambda_i t\}P^{\dagger}(\lambda_i) = 0 \tag{13}$$

since $|P(\lambda_i)| = 0$. Thus, denoting by v_i an arbitrary column of $P^{\dagger}(\lambda_i)$, we know that $\exp\{\lambda_i t\}v_i$ is a solution of the homogeneous system

$$P(D)x = 0 \tag{14}$$

We will assume that all of the characteristic roots (the λ_is) of our differential equation system are distinct.[13] In this case, the general solution to $P(D)x(t) = 0$ is:

$$x^c(t) = \sum_{i=1}^{n} \eta_i e^{\lambda_i t} P_j^{\dagger}(\lambda_i) \tag{15}$$

for arbitrary constants η, where λ_i is the i^{th} root of $|P(\lambda)|$ and $P_j^{\dagger}(\lambda_i)$ is the j^{th} column of $P^{\dagger}(D)$, the

[12]This is just expressing the determinant through expansion by cofactors. (Remember that an expansion by alien cofactors is null.) Of course when the inverse exists $P^{\dagger}(\lambda) = P(\lambda)^{-1}|P(\lambda)|$. However, we will care most about the case when the inverse does not exist.

[13]In general, since repeated roots are not robust in the sense that they disappear with small changes in model parameters, we are not very interested in the case of repeated roots. However, suppose there are k distinct roots $\lambda_i (i = 1, \ldots, k)$, each with multiplicity ω_i. In a manner similar to a single equation, we have the general solution of nonhomogeneous system (3) as follows:

$$\mathbf{x}^g(t) = \mathbf{x}^p(t) + \sum_{i=1}^{k} \sum_{s_i=0}^{\omega_i - 1} c_{is_i} v_{s_i} t^{s_i} \exp\{\lambda_i t\}$$

where $\mathbf{x}^p(t)$ is a particular solution of (3), c_{is_i} stands for an arbitrary scalar and $v_{s_i}(s_i = 0, 1, \ldots, \omega_i - 1)$ are linearly independent column vectors of $P^{\dagger}(\lambda_i)$. If λ_i is a multiple root with multiplicity w_i, then there exist w_i linearly independent columns in $P^{\dagger}(\lambda_i)$(at least in the cases that we will consider; see Murata 3.2 for specific restrictions, esp. Th.6 for some details).

adjoint matrix of $P(D)$, with λ_i in the place of D.[14] If the original system was non-homogeneous, then the general solution can now be written as

$$x^g(t) = x^p(t) + \sum_{i=1}^{n} \eta_i e^{\lambda_i t} P_j^\dagger(\lambda_i) \tag{16}$$

and the unique definite solution can be found by solving for the arbitrary constants η_i using appropriate boundary conditions.

$$x^d(t) = x^p(t) + \sum_{i=1}^{n} c_i v_i \exp\{\lambda_i t\} \tag{17}$$

where v_i satisfies $P(\lambda_i)\, v_i = 0$.[15]

5 Stability

We call a system stable iff the complementary solution $x^c(t)$ must approach zero over time. Equivalently, the general solution approaches a particular solution determined by $u(t)$. From (5) we see immediately that this is only assured when the real parts of all the λ_j are negative.[16] Necessary and sufficient conditions for this are known (see Murata 1977 on the modified Ruth-Hurwitz conditions). Economists are most often interested in analytically solvable models, which practically speaking restricts us to models generating three or fewer roots.

5.1 Single Root

The characteristic equation is $\lambda + a_0 = 0$, so $\lambda = -a_0$. We have stability iff $a_0 > 0$.

5.2 Two roots:

The characteristic equation is $\lambda^2 + a_1\lambda + a_0 = 0$. The roots can be found using the quadratic equation. Stability obtains iff the real parts of the characteristic roots are negative, which is the case iff $a_1, a_0 > 0$.

[14] You will generally be able to choose the columns of the adjoint matrix $P^\dagger(\lambda_i)$ arbitrarily: due to their linear dependence this will only change the constants η in an offsetting manner and will have no effect on the definite solution. However, it is possible to get a zero vector in $P^\dagger(\lambda_i)$, and this (although still linearly dependent with the other columns) should obviously not be used.

[15] Equivalently, if we have a first order system $\dot{\mathbf{x}} = A\mathbf{x} + u$, then $(\lambda_i I - A)v_i = 0$. I.e., the v_i are characteristic vectors of A.

[16] Remember, we are only considering the case of distinct roots. Let $\lambda_j = a_j + ib_j$. Then

$$\begin{aligned} \exp\{\lambda_j t\} &= \exp\{a_j t\} \exp\{ib_j t\} \\ &= \exp\{a_j t\}(\cos b_j t + i \sin b_j t) \end{aligned}$$

Since $\cos b_j t + i \sin b_j t$ is cyclical with constant amplitude, $\exp\{\lambda_j t\}$ converges to zero only if $\exp\{a_j t\}$ does. That is, we need $a_j < 0$ for stability.

Often, however, we hope the outcome will be a saddle-path: one root positive and one root negative. This is the case when $a_0 < 0$.

5.3 Three roots:

The characteristic equation is $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. The roots can be found in various ways. The discussion in Beavis and Dobbs 160–162 is useful for this case.

6 Linear Approximation Systems

If at any time t we take a first order Taylor approximation to $f[x(t)]$ around a point $\bar{x}$, we get

$$f(x_t) - f(\bar{x}) \approx A(x_t - \bar{x})$$

where A is now the Jacobian matrix of f evaluated at $\bar{x}$, i.e., the ij-th element of A is $\partial f_i(\bar{x})/\partial x_j$. Now suppose that $\dot{x}_t = f(x_t)$ and $\bar{x}_t = f(\bar{x}_t)$ and define $\delta x(t) \equiv x(t) - \bar{x}(t)$. This gives us at each time t

$$\begin{aligned} \dot{\delta x}_t &\equiv \dot{x}_t - \dot{\bar{x}}_t \\ &= f(x_t) - f(\bar{x}_t) \\ &\approx A_t(x_t - \bar{x}_t) \\ &= A_t \delta x_t \end{aligned}$$

We will choose $\bar{x}(t)$ so that $f[\bar{x}(t)] = 0$. That is, we take our linear approximation around a stationary state of the original system. Then we need not worry about changes in A, since $\bar{x}(t)$ is constant over time, and we can focus on the following differential equation system.

$$\dot{\delta x} = A\delta x$$

Analysis of this system can be extremely informative. Stability of the resulting "linear approximation system" is sufficient for the (local) stability of the original system! (Note: the converse is not true, i.e., a stable system may not have a stable linear approximation system. Thus it is not really appropriate to use conditions stability of the linear approximation system in comparative statics arguments based on Samuelson's Correspondence Principle.)

7 Summary

Economists often work with linear approximation systems. Suppose that $\dot{x}(t) = f[x(t)]$ and define $\delta x(t) \equiv x(t) - \bar{x}$ where $f(\bar{x}) = 0$. The linear approximation system $\dot{\delta x}(t) = A\delta x(t)$, a first order differential equation system, has a general solution $\delta x(t) = \exp\{At\}\eta$.

Here A is the Jacobian $f_x(\bar{x})$, and $\eta = \delta x^o$ if we are given the initial condition $\delta x(0) = \delta x^o$. Since finding $\exp\{At\}$ is often painful, we take advantage of the following observation. Take any eigenvalue λ and associated eigenvector κ of the Jacobian matrix (so that $A\kappa = \lambda\kappa$), then $\delta x = \kappa \exp\{\lambda t\}$ is a solution.

The adjoint matrix technique is just an elaboration on this observation. Consider a homogeneous system of linear differential equations (not necessarily first-order):

$$P(D)\delta x(t) = 0$$

The characteristic equation of our differential equation system is

$$|P(\lambda)| = 0,$$

where $\mid P(\lambda) \mid$ is an n^{th} order polynomial in the variable λ. The roots of this equation are called the characteristic roots of the differential equation system, and we assume they are all distinct. Let $P^{\dagger}(\lambda)$ denote the adjoint of $P(\lambda)$. Then choosing any root λ_i of the characteristic equation will give us

$$P(D)\exp\{\lambda_i t\}P^{\dagger}(\lambda_i) = 0$$

since $|P(\lambda_i)| = 0$. Thus, denoting by $P_j^{\dagger}(\lambda_i)$ the (arbitrarily chosen) j^{th} column of $P^{\dagger}(\lambda_i)$,we know that $\exp\{\lambda_i t\}P_j^{\dagger}(\lambda_i)$ is a solution of the homogeneous system. The general solution to $P(D)\delta x(t) = 0$ is:

$$\delta x(t) = \eta_i e^{\lambda_i} P_j^{\dagger}(\lambda_i)$$

for arbitrary constants η. The unique definite solution can be found by solving for the arbitrary constants η_i using appropriate boundary conditions.

The system is stable iff $\delta x(t)$ approaches zero over time. Stability therefore requires that all characteristic roots have negative real parts.

So we basically have the following steps:

1. Formulate the dynamic structural model
2. Linearize the model around a steady state
3. Express the linearized model as $P(D)\delta x = 0$
4. Find the characteristic roots of the system
5. Solve the system using the adjoint matrix technique
6. Solve for the η_i using appropriate boundary conditions

Exercises

1. a. Solve $\dot{x}(t) = 5x(t)$ given $x(0) = 2$; and $\dot{x}(t) = -3x(t)$ given $x(0) = 20$. b. Show that your solutions work by differentiating them with respect to t. c. Graph your solutions for $t = 1, 2, \ldots, 10$. d. Make up four similar problems, two with $A > 0$ and two with $A < 0$, solve them, and graph your solutions. e. What importance do you now place on the sign of A in such problems?

2. Show that if $\phi^1(t)$ and $\phi^2(t)$ are solutions to the a linear homogeneous system of differential equations then so is any weighted sum of these two solutions. [Hint: Just differentiate $\phi(t) = \eta_1\phi^1(t) + \eta_2\phi^2(t)$ and inspect your result, paying attention to the definition of a solution.]

3. Show equation (5) is a solution to equation (3) by differentiation.[17] Hint: do not forget that the fundamental theorem of calculus implies $(d/dt)\int f(s)ds = f(t)$.

4. Solve $\dot{x} = 3x + 6$ given $x^o = 5, \dot{x} = -18x + 9$ given $x^0 = 0$; and $\dot{x} = 5$ given $x^0 = 2$.

   (Note that the forcing functions are of the form $u(t) = k$ where k is a constant; therefore, guess that your particular solution has this form too.) Check your solutions, and show all of your work.

[17] For this homework, you may work with the univariate case so that x and A are scalars. When $\boldsymbol{x}$ is a vector and $\boldsymbol{A}$ is a matrix, the procedure is similar. To see this, recall that

$$\exp\{A\} \equiv I + A + \frac{A^2}{2!} + \ldots = \sum \frac{A_i}{i!}$$

so,

$$\begin{aligned}
\frac{d}{dt}\exp\{At\} &= \lim_{h\to 0} \frac{\exp\{A(t+h)\} - \exp\{At\}}{h} \\
&= \exp\{At\} \lim_{h\to 0} \frac{Ah - I}{h} && (*) \\
&= \exp\{At\}\, A \\
&= A\exp\{At\} && (**)
\end{aligned}$$

* $\exp\{A+B\} = \exp\{A\}\exp\{B\}$ if and only if $AB = BA$ (Proof in J.E. Woods, pg. 119). Note: this implies the matrix exponential is invertible since $\exp\{A - A\} = I$.
** Note that A commutes with each term in $\exp\{At\}$.

5. It is clear that (5) is a solution, but how do we know that it is the *general* solution, the representation of all possible solutions? [Hint: consider the first order case, and peruse the theorems in this chapter.]

6. Suppose a linear system of constant coefficients has a characteristic equation $\lambda^2 + a_1\lambda + a_0 = 0$. Show that the characteristic roots are opposite in sign iff $a_0 < 0$, and show that they have negative real parts iff $a_1, a_0 > 0$.

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