> [O]ur telescopic faculty is defective .... [W]e ... see future pleasures, as it were, on a diminished scale .... [P]eople distribute their resources between the present, the near future, and the remote future on the basis of a wholly irrational preference .... The inevitable result is that efforts directed towards the remote future are starved relatively to those directed to the near future, while those in turn are starved relatively to efforts directed towards the present.
> Pigou, Arthur Cecil. 1920. The Economics of Welfare (London: Macmillan), p.25.

This reasonable view of Pigou's is precisely what we will reject in our exploration of the application of dynamic programming methods to the solution of economic problems.

1 Markov Chains

Markov chains often arise in dynamic optimization problems.

Definition 1.1 (Stochastic Process) A ***stochastic process*** is a sequence of random vectors.

We will index the sequence with the integers, which is appropriate for discrete time modeling.

Definition 1.2 (Markov Process) A stochastic process $\{x_t\}$ has the ***Markov Property*** if

$$\mathbf{P}\{x_{t+1} \mid x_t, x_{t-1}, \ldots, x_0\} = \mathbf{P}\{x_{t+1} \mid x_t\}$$

for any index t. A stochastic process with the Markov property is called a ***Markov process***.

Definition 1.3 (Transition Matrix) A vector of non-negative real numbers that sum to unity is called a ***probability vector***. A ***probability matrix*** is a matrix where each row is a probability vector.[1] A square probability matrix is called a ***transition matrix***.

Definition 1.4 (Markov Chain) A time-invariant ***Markov chain*** is a discrete time Markov process with constant transition probabilities. We characterize a Markov chain by three objects: a state-space vector $\bar{\boldsymbol{x}}$ (giving the possible states of the process), a probability vector $\boldsymbol{\pi}_0$ whose elements are the probabilities of being in each state initially, and a transition matrix $\boldsymbol{P}$ whose element p_{rk} is the probabilities of transitioning from state r to state k.

Since p_{rk} is the probability of being in state $\bar{x}_k$ next period given state $\bar{x}_r$ this period, we can compute the probability of being in state $\bar{x}_k$ two periods from now given state $\bar{x}_r$

[1] This is sometimes called a *right* probability matrix, where each *column* of a left probability matrix is a probability vector.

this period as follows:

$$\begin{aligned}\mathbf{P}\{x_{t+2}=\bar{x}_k \mid x_t=\bar{x}_j\} &= \sum_{\nu=1}^{N}\mathbf{P}\{x_{t+2}=\bar{x}_k \mid x_{t+1}=\bar{x}_\nu\}\mathbf{P}\{x_{t+1}=\bar{x}_\nu \mid x_t=\bar{x}_r\} \\ &= \sum_{\nu=1}^{N} p_{\nu k}p_{r\nu} = \sum_{\nu=1}^{N} p_{r\nu}p_{\nu k} = \boldsymbol{P}_{r\cdot}\boldsymbol{P}_{\cdot k}\end{aligned} \tag{1}$$

which is the r,k-th element of $\boldsymbol{P}^2$.

So element r,k of the matrix $\boldsymbol{P}^2$ gives the probability of moving from state $\bar{x}_r$ to state $\bar{x}_k$ in two periods. This reasoning is easily extended across any number of periods. So we can conclude that the unconditional probability vector for x_t is

$$\mathbf{P}\{\boldsymbol{x}_t\} = \boldsymbol{\pi}_0^{\tau}\boldsymbol{P}^t \tag{2}$$

Here we use the shorthand that $\mathbf{P}\{\boldsymbol{x}_t\}$ has as its r-th element the unconditional probability that $x_t = \bar{x}_r$.

2 Expected Utility

We begin with an exposition based on ljungqvist_sargent p.45: compute the expected lifetime utility V of an infinitely lived consumer:

$$V(c_t,\lambda_t) = \mathcal{E}_t\sum_{\tau=t}^{\infty}\beta^{t-\tau}u(c_\tau) \tag{3}$$

For the moment, we are not attempting any optimization: we specify that $c_{t+1} = \lambda_{t+1}c_t$, where λ_t be an n-state Markov process with possible values $\boldsymbol{\lambda}$ and associated transition matrix P, $0 \le \beta \le 1$, and $\mathcal{E}_t$ is the expectation condition on information available at time t (including the values of c and λ in periods t or earlier). Note that our definition of V implies that we can restate this recursively

$$\begin{aligned}V(c_t,\lambda_t) &= \mathcal{E}_t\sum_{\tau=0}^{\infty}\beta^{\tau}u(c_{t+\tau}) \\ &= u(c_t) + \mathcal{E}_t\sum_{\tau=0}^{\infty}\beta^{\tau+1}u(c_{t+1+\tau}) \\ &= u(c_t) + \mathcal{E}_t\beta\mathcal{E}_{t+1}\sum_{\tau=0}^{\infty}\beta^{\tau}u(c_{t+1+\tau}) \\ &= u(c_t) + \mathcal{E}_t\beta V(c_{t+1},\lambda_{t+1})\end{aligned} \tag{4}$$

Suppose there is a solution of the form $V(c_t,\lambda_t) = u(c_t)w(\lambda_t)$. (We will soon consider a situation where this is the case.) This would imply that

$$u(c_t)w(\lambda_t) = u(c_t) + \beta \mathcal{E}_t u(c_{t+1}) w(\lambda_{t+1}) \tag{5}$$

or

$$w(\lambda_t) = 1 + \beta \mathcal{E}_t [u(c_{t+1})/u(c_t)] w(\lambda_{t+1}) \tag{6}$$

So if $u(c_{t+1})/u(c_t)$ can be written as a simple function of λ_{t+1}, we end up with an easily solved linear system.

For example, if utility is of the constant relative risk aversion functional form, so that $u(c) = c^{1-\gamma}/(1-\gamma)$, then $u(c_{t+1})/u(c_t) = (c_{t+1}/c_t)^{1-\gamma} = \lambda_{t+1}^{1-\gamma}$, and we end up with the following system

$$w(\lambda_t) = 1 + \beta \mathcal{E}_t \lambda_{t+1}^{1-\gamma} w(\lambda_{t+1}) \tag{7}$$

or

$$\boldsymbol{w} = \mathbf{1} + \beta \boldsymbol{P} \mathrm{diag}(\boldsymbol{\lambda})^{1-\gamma} \boldsymbol{w} \tag{8}$$

which has the solution

$$\boldsymbol{w} = \left(\boldsymbol{I} - \beta \boldsymbol{P} \mathrm{diag}(\boldsymbol{\lambda})^{1-\gamma}\right)^{-1} \mathbf{1} \tag{9}$$

3 Optimization: Current versus Future Consumption

We begin with a particularly simple problem. A consumer has assets k_t from which to consume this period, t, and next period, $t+1$. After consuming c_t this period, the consumer carries forward $k_t - c_t$ in savings, which becomes

$$k_{t+1} = R(k_t - c_t) \tag{10}$$

next period, based on the gross interest rate R. More generally, we can allow $k_{t+1} = g(k_t, c_t)$ subject to mild restrictions on g.

Let $V_{t+1}(k_{t+1})$ represent the consumer's future utility (i.e., next period payoff) from that k_{t+1} in wealth. At this point we do not know anything else about the function V. Looking forward from period t, the consumer discounts this payoff to $\beta V_{t+1}(k_{t+1})$. Here β is a *discount factor*, with $0 < \beta < 1$ representing the idea that a current payoff is worth more than a future payoff, ceteris paribus. We can represent the consumer's objective function in period t as

$$u(c_t) + \beta V_{t+1}\big(k_{t+1}\big) \tag{11}$$

The consumer will choose c_t so as to make the objective function as large as possible, subject to the budget constraint (10).

Substituting the budget constraint that $k_{t+1} = R(k_t - c_t)$ into the objective function, we can represent the consumer's objective function in period t as

$$u(c_t) + \beta V_{t+1}\big(R(k_t - c_t)\big) \tag{12}$$

The consumer will choose c_t so as to make the objective function as large as possible. The optimal value of the objective function depends on the consumer's initial wealth, so let us write this optimal value as $V_t(k_t)$. That is, we define

$$V_t(k_t) \equiv \max_{c_t}\{u(c_t) + \beta V_{t+1}\big(R(k_t - c_t)\big)\} \tag{13}$$

We will call $V_t(k_t)$ the value of k_t. Clearly V_t is just an indirect utility function. As an *indirect* utility function, it does not depend on c_t. We will refer to this characterization of the indirect utility function as a Bellman equation.

A consumer seeking the optimal value of the objective function must make the best trade-off between the utility of current consumption, $u(c_t)$, and the reduction in future utility (due to lower wealth) caused by higher consumption today. That is, an optimizing consumer must satisfy the first-order condition

$$u' - R\beta V'_{t+1} = 0 \tag{14}$$

This first-order condition nicely characterizex a necessary condition for optimization: the benefit from one more unit of consumption, u', must just balance the cost of having R fewer units to consumer next period, $R\beta V'$. We can think of the Bellman equation as the result of solving (14) for the optimal level of consumption, $\hat{c}$, and plugging that into (12).

Ideally, we would put this information to work do determine consumption as a function of the current state. We could then simulate the behavior of the economy, starting from an initial state k_0, as follows:

- determine $c_t = h(k_t)$
- determine $k_{t+1} = g(k_t, c_t)$

4 An Application to Optimal Growth

The following is based on brock_mirman1972 and ljungqvist_sargent p.33. A planner wants to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{15}$$

subject to

$$k_{t+1} = Ak_t^\alpha - c_t \tag{16}$$

Bellman equation:

$$V\big(k_t\big) = \max_{c_t}\big\{\ln(c_t) + \beta V_{t+1}\big(Ak_t^\alpha - c_t\big)\big\} \tag{17}$$

First order condition:

$$1/c_t = \beta V'_{t+1} \tag{18}$$

which implies that

$$V'_{t+1} = 1/c_t\beta \tag{19}$$

Envelope condition:

$$V_t' = \alpha A k^{\alpha-1} \beta V_{t+1}' \tag{20}$$

into which we substitute from the FOC to get

$$V_t' = \alpha A k_t^{\alpha-1}/c_t \tag{21}$$

so that

$$V_{t+1}' = \alpha A k_{t+1}^{\alpha-1}/c_{t+1} \tag{22}$$

which we substitute back into the FOC to get

$$1/c_t = \beta \alpha A k_{t+1}^{\alpha-1}/c_{t+1} \tag{23}$$

4.1 Computing the Solution

ljungqvist_sargent p.32 discuss three methods for solving dynamic programs.

Guess and verify In a few cases with unique solutions, we may be lucky enough to be able to guess the solution and verify it.

Value function iteration Recall that a function is just a collection of ordered pairs. Computationally function iteration is generally going to mean characterizing the function with a finite set of points.

Letting V_i now represent our value function at the i-th iteration, we would have

$$V_{i+1}(k) = \max_c \left\{ u(c) + \beta V_i\big(g(k,c)\big) \right\} \tag{24}$$

for a given value of k.

Policy function iteration Policy function iteration is often faster than value function iteration. A common approach is Howard's improvement algorithm. We initialize the algorithm by finding a feasible initial policy h_0, and then iterate to convergence over the following two steps

- compute value of current policy: $V_{h_i} = \sum_{t=0}^{\infty} \beta^t u(h_i)$ subject to the transition equation $k_{t+1} = g(k_t, c_t)$.
- compute a new policy h_{i+1} as the solution to the following two period problem:

$$\max_c u(c) + \beta V^{h_i}\big(g(k_t, c_t)\big)$$

4.2 Application to our growth problem

Value function iteration Initialize $V_0(k) = 0$.

Iterative step 1:

$$V_1(k) = \max_c \left\{ \ln(c) + \beta V_0\big(Ak^\alpha - c\big) \right\} \tag{25}$$

In this first step, we impose that next period's capital stock cannot be negative, so we get $c = Ak^\alpha$ and thus

$$V_1(k) = \ln\left(Ak^\alpha\right) = \ln A + \alpha \ln k \tag{26}$$

Iterative step 2:

$$V_2(k) = \max_c \left\{ \ln(c) + \beta[\ln A + \alpha \ln(Ak^\alpha - c)] \right\} \tag{27}$$

FOC:

$$1/c - \alpha\beta/(Ak^\alpha - c) = 0 \tag{28}$$

$$c = Ak^\alpha/(1 + \alpha\beta) \tag{29}$$

with the implied value function

$$\begin{aligned} V_2(k) &= \ln\left(Ak^\alpha/(1+\alpha\beta)\right) + \beta[\ln A + \alpha \ln(\alpha\beta Ak^\alpha/(1+\alpha\beta))] \\ &= \ln\left(A/(1+\alpha\beta)\right) + \beta \ln A + \alpha\beta \ln(\alpha\beta A/(1+\alpha\beta)) + \alpha(1+\alpha\beta)\ln(k) \end{aligned} \tag{30}$$

Iterative step 3:

$$V_3(k) = \max_c \left\{ \ln(c) + \beta[\text{constant} + \alpha(1+\alpha\beta)\ln(Ak^\alpha - c)] \right\} \tag{31}$$

FOC:

$$1/c - \alpha\beta(1+\alpha\beta)/(Ak^\alpha - c) = 0 \tag{32}$$

$$c = Ak^\alpha/(1 + \alpha\beta + \alpha^2\beta^2) \tag{33}$$

At this point we can see that c is iterating toward $c^* = (1 - \alpha\beta)Ak^\alpha$, and that the value function will be of the form

$$V(k) = \Pi_0 + \Pi_1 \ln(k) \tag{34}$$

We can also see a pattern developing for the value function, which determines these two constants. But let us use this opportunity to apply the method of undetermined coefficients, the second method we will illustrate for determining the value function and optimal "policy". Here Π_0 and Π_1 are undetermined coefficients. The associated Bellman equation is therefore

$$\Pi_0 + \Pi_1 \ln(k) = \max_c \{\ln(c) + \beta[\Pi_0 + \Pi_1 \ln(Ak^\alpha - c)]\} \tag{35}$$

Look at the associated FOC:

$$1/c = \beta\Pi_1/(Ak^\alpha - c) \tag{36}$$

or

$$c = Ak^{\alpha}/(1 + \beta\Pi_1) \tag{37}$$

Plug this back into the Bellman equation to get the identity

$$\begin{aligned}\Pi_0 + \Pi_1 \ln(k) &= \ln(Ak^{\alpha}/(1 + \beta\Pi_1)) + \beta[\Pi_0 + \Pi_1 \ln(\beta\Pi_1 Ak^{\alpha}/(1 + \beta\Pi_1))] \\ &= \ln(A/(1 + \beta\Pi_1)) + \beta\Pi_0 + \beta\Pi_1 \ln(\beta\Pi_1 A/(1 + \beta\Pi_1))]\alpha(1 + \beta\Pi_1)\ln(k)\end{aligned} \tag{38}$$

This is an identity: it must hold for *any* value of k. So we must have

$$\Pi_1 = \alpha(1 + \beta\Pi_1) \tag{39}$$

or

$$\Pi_1 = \alpha/(1 - \alpha\beta) \tag{40}$$

Exercise 1
Show that

$$\Pi_0 = \frac{1}{1-\beta}\left(\ln A(1 - \alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta A\right) \tag{41}$$

OK, let us return to a consideration of our optimal consumption:

$$c_t^* = (1 - \alpha\beta)Ak_t^{\alpha} \tag{42}$$

so that

$$k_{t+1}^* = \alpha\beta Ak_t^{\alpha} \tag{43}$$

So the optimal policy is for capital to evolve according to this nonlinear, first-order difference equation. Convergence is ensured by the standard presumption that $0 < \alpha < 1$, so we can talk of a steady state where

$$k_{ss}^* = \alpha\beta Ak_{ss}^{\alpha} \tag{44}$$

or

$$k_{ss}^* = \big(\alpha\beta A\big)^{1/(1-\alpha)} \tag{45}$$

5 An Application to Consumption

Consider an infinitely lived consumer who retains undiminished capacity to enjoy consumption, as represented by the unchanging untility function u. We capture this in the following objective function:

$$\max \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{46}$$

The consumer is subject to the transition equation

$$k_{t+1} = R_t(k_t + Y_t - c_t) \tag{47}$$

which tells us how wealth evolves over time. (We think of Y_t as disposable income, and $R = (1 + r)$ is the gross real interest rate.) The crucial point is that today's consumption affects tomorrow's wealth.

We want to characterize the value of consuming optimally today and in the future, as this depends on our current wealth. The key observation is that, if we are behaving optimally, the value of our wealth today can be broken into the benefit derived from current consumption and the benefit derived from next period's wealth. Since this is an infinite horizon problem, we get a particularly simple representation: the function V is not time dependent.

$$V(k_t, \dots) = \max_{c_t}\{u(c_t) + \beta \mathcal{E}_t V(k_{t+1}, \dots)\} \tag{48}$$

The best we can do today is to pick consumption so as to make the optimal trade-off between current utility and future utility. This trade-off shows up in the effect of consumption on future wealth, as captured by the transition equation. The Bellman equation is our representation of this trade-off:

$$V(k_t, \dots) = \max_{c_t}\{u(c_t) + \beta \mathcal{E}_t V[R_t(k_t + Y_t - c_t), \dots]\} \tag{49}$$

Note how we substituted for future wealth from the transition equation in order to obtain the Bellman equation. We will associate a first order condition and an envelope condition with the Bellman equation.

Note that on the right hand side of the Bellman equation we have an *indirect* objective function: the right hand side is *not* a function of c. The first order condition is associated with the terms within the braces: it represents a condition that must hold given that consumption has been set to its optimal level.

$$u'(c_t) - \beta \mathcal{E}_t R_t \frac{\partial V}{\partial k_{t+1}} = 0 \tag{50}$$

Again, we are *not* differentiating both sides with respect to c_t to get this first order condition. Rather is it the term *within the braces* that we are differentiating. The FOC just says that consuming wealth today or getting the expected payoff from saving it must be equally good options at the margin.

Recall the envelope theorem says that, at an optimum, the partial response of the objective function to a parameter change is equal the change in its optimal value. That is, we can safely ignore the changes in the optimal value of the choice variable. We will be interested in the effect of a change in wealth. The envelope condition is

$$\frac{\partial V}{\partial k_t} = \beta \mathcal{E}_t R_t \frac{\partial V}{\partial k_{t+1}} \tag{51}$$

We see we can replace the right hand side of the envelope condition (51) by substituting from the first order condition (50).

$$\frac{\partial V}{\partial k_t} \overset{FOC}{=} u'(c_t) \tag{52}$$

So the value of an additional unit of wealth today equals the marginal utility of current consumption.

This observation allows us to replace the right hand side of the first order condition, producing a relationship that links the choice variable in successive periods. This relationship is known as the Euler equation.

$$u'(c_t) = \beta \mathcal{E}_t R_t u'(c_{t+1}) \tag{53}$$

The interpretation of the Euler equation is very natural. At the optimum, you must be indifferent between consuming an additional unit of wealth today and waiting for the expected payoff from consuming it tomorrow.

Note what we have done and what we have not. We have characterized an important property of the optimal consumption path—one that can be used in empirical tests. However, we have not solved for the optimal level of consumption. More precisely, we have not solved for the "policy function" that would express c_t as a function of k_t.

5.1 The Euler Equation: Special Cases

Hall (1978 JPE) considers the case where income is the only source of uncertainty. When the interest rate is non-stochastic we have the relationship

$$\mathcal{E}_t u'(c_{t+1}) = \frac{1}{\beta R_t} u'(c_t) \tag{54}$$

emphasized by Hall (1978 JPE). This suggests the regression equation

$$u'(c_{t+1}) = \frac{1}{\beta R_t} u'(c_t) + \varepsilon_t \tag{55}$$

where $\mathcal{E}_t \varepsilon_{t+1} = 0$.

That is, except for a trend contributed by β and R, marginal utility is expected to be the same this period and next period. Or as Hall (1978 JPE, p.971) puts it, "Optimization on the part of consumers is shown to imply that the marginal utility of consumption evolves according to a random walk with trend." The key implication that Hall will pursue is that current consumption is a sufficient statistic for future consumption: no other current variables should help in forecasting future consumption. To put it another way (p.972): "In a forecasting model, consumption should be treated as an exognenous variable."

Hall (p.971) adds, "To a reasonable approximation, consumption itself should evolve in the same way." That is, if we regress consumption on its past values and other past variables, the only significant variable should be the one-period lag of consumption. Note that this addresses *past* variables; it does not rule out the importance of current income if that is added to the regression. Why? Because current income can contain new information about permanent income.

5.2 Hall (1978 JPE)

5.2.1 Constant Elasticity of Substitution

Let $u(c) = c^{(\sigma-1)/\sigma}$ implying $u'(c) = \frac{\sigma-1}{\sigma}c^{-1/\sigma}$. The Euler equation with non-stochastic R then implies

$$\mathcal{E}_t c_{t+1}^{-1/\sigma} = \frac{1}{\beta R_t} c_t^{-1/\sigma} \tag{56}$$

Hall (1978 JPE) considers three cases: $\sigma = 2, 1, -1$.

Note that if $\sigma = -1$ we have $u(c) = c^2$ so that $u'(c) = 2c$. This is the case of quadratic utility, although we usually turn to a slight modification in order to get declining marginal utility of consumption.

5.2.2 Quadratic Utility

Let $u(c) = -(\bar{c} - c)^2/2$ implying $u'(c) = \bar{c} - c$.

$$\bar{c} - c_t = \beta \mathcal{E}_t R_t(\bar{c} - c_{t+1}) \tag{57}$$

Note $u'(c) = (\bar{c} - c)$. Here $\bar{c}$ is Bliss Level. Solving for $\mathcal{E}_t c_{t+1}$ we get

$$\beta \mathcal{E}_t R_t c_{t+1} = \bar{c}(\beta R - 1) + c_t \tag{58}$$

Consider the case where R is constant.

$$\begin{aligned} \mathcal{E}_t c_{t+1} &= \bar{c}(1 - 1/\beta R) + (1/\beta R)c_t \\ &= b_o + b_1 c_t \end{aligned} \tag{59}$$

This suggests a natural regression equation, if we can get data on c_t.

$$c_{t+1} = b_o + b_1 c_t + \epsilon_t \tag{60}$$

This is Hall's result that consumption follows a martingale with drift.[2] This suggests that c_t tells us everything useful that can be know about c_{t+1} at time t. *Adding* other variables known at time t shouldn't help predict c_{t+1}. For example, it shouldn't help to *add* income or government expenditure, or even expected future income or expected government expenditure.

[2] A true martingale x_t satisfies $\mathcal{E}_t \Delta x_{t+1} = 0$; consider for example a random walk process. So we use the term here more loosely to accommodate any process where the current value fully determines the expected future value. In the special case considered below, where $\beta = R$, consumption follows a true martingale.

5.2.3 Special Case: $\beta = R^{-1}$

From the budget constraint, the present value of consumption must equal the present value of income plus current wealth.

$$\mathcal{E}_t \sum_{\tau=t}^{\infty} \frac{1}{R^{\tau-t}} c_\tau = k_t + \mathcal{E}_t \sum_{\tau=t}^{\infty} \frac{1}{R^{\tau-t}} Y_\tau \tag{61}$$

Looking at (59) and considering the special case where the rate of time preference equals the rate of interest, $\mathcal{E}_t c_{t+1} = c_t$. So, recalling $\sum_{i=0}^{\infty} a^i = 1/(1-a)$, we can rewrite the budget constraint as

$$c_t = \frac{R-1}{R}\left[k_t + \mathcal{E}_t \sum_{\tau=t}^{\infty} \frac{1}{R^{\tau-t}} Y_\tau\right] \tag{62}$$

This in turn implies

$$c_{t+1} = \frac{R-1}{R}\left[k_{t+1} + \mathcal{E}_{t+1} \sum_{\tau=t+1}^{\infty} \frac{1}{R^{\tau-t-1}} Y_\tau\right] \tag{63}$$

$$\mathcal{E}_t c_{t+1} = \mathcal{E}_t \frac{R-1}{R}\left[k_{t+1} + \mathcal{E}_{t+1} \sum_{\tau=t+1}^{\infty} \frac{1}{R^{\tau-t-1}} Y_\tau\right] \tag{64}$$

So

$$c_{t+1} - \mathcal{E}_t c_{t+1} = \frac{R-1}{R}[k_{t+1} - \mathcal{E}_t k_{t+1}] + \frac{R-1}{R}(\mathcal{E}_{t+1} - \mathcal{E}_t) \sum_{\tau=t+1}^{\infty} \frac{1}{R^{\tau-t-1}} Y_\tau \tag{65}$$

But wealth is a predetermined variable, so $\mathcal{E}_t k_{t+1} = R(k_t + Y_t - c_t) = k_{t+1}$, and we have seen that $\mathcal{E}_t c_{t+1} = c_t$.

$$c_{t+1} - c_t = \frac{R-1}{R}(\mathcal{E}_{t+1} - \mathcal{E}_t) \sum_{\tau=t+1}^{\infty} \frac{1}{R^{\tau-t-1}} Y_\tau \tag{66}$$

This is just an expression for the change in the level of consumption. The change in consumption is the annuity value of the change in the discounted present value of expected future income.

So we should not see any change in consumption in response to fully anticipated changes in income: that is, there should be no excess sensitivity. Also, we should see a full response of consumption to changes in permanent income: there should be no excess smoothness. We will look at some empirical work testing these hypotheses, but first we will reconsider Aschauer's definition of consumption.

5.3 Aschauer (1985 AER): $c_t^* = c_t + \theta G_t$

Bailey (1971) first (?) proposed that a unit of publically provided goods and services can be valued as θ units of private consumption, with $0 < \theta < 1$. In this case, we let

$c^* = c + \theta G$ represent "effective" consumption, and our objective function becomes

$$\max \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^*) \tag{67}$$

Note $dc_t^*/dc_t = 1$. Convince yourself that the transition function (47) is equivalent to Aschauer's eqn 3. The Bellman equation is therefore

$$V(k_t, \dots) = \max_{c_t}\{u(c_t^*) + \beta \mathcal{E}_t V[R_t(k_t + Y_t - c_t), \dots]\} \tag{68}$$

with associated Euler equation

$$u'(c_t^*) = \beta \mathcal{E}_t R_t u'(c_{t+1}^*) \tag{69}$$

In the case of quadratic utility, we find (following our analysis of Hall above)

$$\begin{aligned} \mathcal{E}_t c_{t+1}^* &= \bar{c}^*(1 - 1/\beta R) + (1/\beta R) c_t^* \\ &= b_o + b_1 c_t^* \end{aligned} \tag{70}$$

Hall (1978 JPE) assumes all utility is derived from direct consumption expenditures: $\theta = 0$.

Aschauer (1985 AER) asks whether C responds to

1. G? i.e., Does part of G substitute for c? Answer: yes, but not much.

2. Tax vs. debt financing? Answer: no.

Aschauer assumes $c_t^* = c_t + \theta G_t$ where $0 < \theta < 1$, so in the quadratic utility case we have

$$\mathcal{E}_t(\overbrace{c_{t+1} + \theta G_{t+1}}^{c_{t+1}^*}) = b_o + b_1(\overbrace{c_t + \theta G_t}^{c_t^*}) \tag{71}$$

$$\mathcal{E}_t c_{t+1} = b_o + b_1 c_t + b_1 \theta G_t - \theta \mathcal{E}_t G_{t+1} \tag{72}$$

We would like to run the regression

$$c_t = b_o + b_1 c_{t-1} + b_1 \theta G_{t-1} - \theta G_t^e + u_t \tag{73}$$

But we don't observe G_t^e. Following fairly standard practice, assume the data generating process for G_t can be modeled as depending on previous spending patterns and deficits.

$$G_t = \gamma + e(L) G_{t-1} + w(L) D_{t-1} + v_t \tag{74}$$

where $v_t \sim$WN so that

$$G_t^e = \gamma + e(L) G_{t-1} + w(L) D_{t-1}$$

Table 1: Aschauer's Table 3: FIML Results for his Equations 15

1948:I to 1981:IV with $n=2, m=1$.		
Constrained	Unconstrained	Hypothesized
$\alpha = 1.370$ (.121)	$\delta = 1.930$ (1.239)	$\delta = 1.068$
$\beta = 1.002$ (.001)	$\beta = .990$ (.002)	$\beta = 1.002$
$\theta = .231$ (.113)	$\eta_1 = -0.26$ (.057)	$\eta_1 = -.093$
$\gamma = 1.308$ (.654)	$\eta_2 = .037$ (.056)	$\eta_2 = .090$
$\varepsilon_1 = 1.404$ (.075)	$\mu_1 = -.029$ (.015)	$\mu_1 = -.014$
$\varepsilon_2 = -.403$ (.075)	$\gamma = 1.278$ (.150)	$\gamma = 1.308$
$\omega_1 = .061$ (.016)	$\varepsilon_1 = 1.442$ (.075)	$\varepsilon_1 = 1.404$
$h_C = 1.340$	$\varepsilon_2 = -.441$ (.075)	$\varepsilon_2 = -.403$
$h_G = .010$	$\omega_1 = .049$ (.018)	$\omega_1 = .061$
$\bar{R}^2_C = .998$	$\bar{R}^2_C = .999$	
$\bar{R}^2_G = .998$	$\bar{R}^2_G = .998$	
$+2\log_e(L_r/L_u) = 4.280$		

α: our b_0, the constant from consumption function
β: our b_1, the coefficient on c_{t-1} from consumption function
γ: the constant from G regression
θ: the substitution coefficient of interest, the weight on G_t in utility function
h_C: Durbin-h for consumption equation
h_G: Durbin-h for G equation

Substituting for G^e_t, we get

$$c_t = \delta + b_1 c_{t-1} + \eta(L)G_{t-1} + \mu(L)D_{t-1} + u_t \tag{75}$$

where

$$\begin{aligned} \delta &= b_o - \theta\gamma \\ \eta_1 &= \theta(b_1 - e_1) \\ \eta_i &= -\theta e_i \qquad i = 2, \ldots, n \\ \mu_j &= -\theta w_j \qquad j = 1, \ldots, m \end{aligned} \tag{76}$$

Estimate (74) and (75) by FIML (imposing cross equation restrictions). Aschauer reports the results in his Tables 2 and 3.

Key results:

θ small but significant;
$b_1 \approx 1$;
the RatEx constraints (76) are satisfied so that deficits only matter by affecting projected G (Ricardian equivalence supported to this extent; c doesn't respond to a change in method of finance.)

6 Portfolio Allocation with a Risk Free Asset

In this case, $R_t = R(1-\omega_t) + Z_t\omega_t$.

- one risk free asset with gross return R
- one risky asset with gross return $Z_t = \bar{Z} + \varepsilon_t$
- new control variable: fraction ω of portfolio in risky asset

Before looking at the implications in a dynamic programming framework, consider a very simple case. At the beginning of the period you have one unit of wealth to allocate among the two assets, and you will derive (increasing, concave) utility by consuming the end of period value.

$$\max_{\omega} \mathcal{E}u\big(\omega Z + (1-\omega)R\big) \tag{77}$$

Time is really irrelevant in this simple version, so we have dropped all time subscripts. The first order condition is

$$\mathcal{E}u' \cdot (Z - R) = 0 \tag{78}$$

Since R is non-stochastic, we can rewrite this as

$$\mathcal{E}u' \cdot Z = R\mathcal{E}u' \tag{79}$$

Recalling that $\mathcal{E}xy = \mathcal{E}x\mathcal{E}y + \mathrm{cov}(x,y)$,

$$\mathcal{E}u' \ \mathcal{E}Z + \mathbf{cov}(u', Z) = R\mathcal{E}u' \tag{80}$$

$$\mathcal{E}Z = R - \frac{\mathbf{cov}(u', Z)}{\mathcal{E}u'} \tag{81}$$

This is the core equation of the consumption CAPM. In this simple setting we expect Z and u' to be negatively correlated, implying that the consumer must be compensated (by a higher average return) if she is to hold the risky asset.

Now let us move to a dynamic setting.

Objective:

$$\max_{\omega,c} \mathcal{E}_0 \sum_{t=0}^{T+1} \beta^t u(c_t) \qquad \text{s.t. } k_{t+1} = (k_t - c_t)[R(1-\omega_t) + Z_t\omega_t] \tag{82}$$

Define the state variable: k_t with transition equation

$$k_t = (k_{t-1} - c_{t-1})[R(1-\omega_{t-1}) + Z_{t-1}\omega_{t-1}] \tag{83}$$

Define the control vector: (c_t, ω_t)

Bellman's Equation:

$$V_t(k_t) = \max_{\omega_t, c_t} \left\{ u(c_t) + \beta \mathcal{E}_t V_{t+1} \{ \overbrace{(k_t - c_t)[R(1-\omega_t) + Z_t\omega_t]}^{k_{t+1}} \} \right\} \tag{84}$$

First order conditions:

$$c: \qquad u'(c_t) - \beta \mathcal{E}_t [\overbrace{R(1-\omega_t) + Z_t\omega_t}^{R_t}] \partial V_{t+1}/\partial k_{t+1} = 0 \tag{85}$$

$$\omega: \qquad \beta \mathcal{E}_t [Z_t - R] \partial V_{t+1}/\partial k_{t+1} = 0 \tag{86}$$

Envelope condition:

$$\begin{aligned} \partial V_t/\partial k_t &= \beta \mathcal{E}_t [R(1-\omega_t) + Z_t\omega_t] \partial V_{t+1}/\partial k_{t+1} \\ &\overset{\text{FOC}}{=} u'(c_t) \end{aligned} \tag{87}$$

Therefore we can rewrite the FOCs as

$$u'(c_t) - \beta \mathcal{E}_t [R(1-\omega_t) + Z_t\omega_t] u'(c_{t+1}) = 0 \tag{88}$$

$$\beta \mathcal{E}_t [Z_t - R] u'(c_{t+1}) = 0 \tag{89}$$

which give us

$$u'(c_t) = \beta R \mathcal{E}_t u'(c_{t+1}) \tag{90}$$

$$u'(c_t) = \beta \mathcal{E}_t Z_t u'(c_{t+1}) \tag{91}$$

The expected payoff to consumption versus saving must be equal at the margin, for both assets.

Comment: equation (90) is familiar to us from Hall (1978). Recall that we learned from Hall that we do not need an explicit solution for consumption to test the theory, since we can test the strong implications of the FOCs. (See Hansen.)

Now recall that

$$\mathcal{E}xy = \mathcal{E}x\mathcal{E}y + \mathrm{cov}(x, y) \tag{92}$$

and reconsider (89).

$$\mathcal{E}_t [Z_t u'(c_{t+1})] - R \mathcal{E}_t u'(c_{t+1}) = 0 \tag{93}$$

$$\mathcal{E}_t Z_t E_t u'(c_{t+1}) + \mathrm{cov}[Z_t, u'(c_{t+1})] = R \mathcal{E}_t u'(c_{t+1}) \tag{94}$$

$$\mathcal{E}_t Z_t = R - \frac{\mathrm{cov}[Z_t, u'(c_{t+1})]}{\mathcal{E}_t u'(c_{t+1})} \tag{95}$$

Equation (95) is know as the consumption CAPM, and it is has often been tested with aggregate data.

Key message: assuming $u'' < 0$, we see that a lower expected payoff from the risky asset is acceptable *if* it tends to have a relatively high payoff when c_t is low. That is, it must serve as a consumption hedge.

Note: (B&F p.508) although the consumption CAPM should hold for each optimizing individual, strong assumptions are required for the use of aggregate data. Typically invoke a representative infinitely lived agent facing a fixed set of assets with a stationary (at least after differencing?) distribution of returns.

HW: Consider the problem with n risky assets:

$$\max_{\omega_i,c} \mathcal{E}_0 \sum_{t=0}^{T+1} \beta^t u(c_t) \qquad \text{s.t. } k_{t+1} = (k_t - c_t)[R(1 - \sum_{i=0}^{n} \omega_{i,t}) + \sum_{i=0}^{n} Z_{i,t}\omega_{i,t}] \tag{96}$$

Show that the consumption CAPM holds for each asset.

Recommended Reading

References

Araujo, A. (1991, September). "The Once But Not Twice Differentiability of the Policy Function." *Econometrica* 59(5), 1383–1393.

Aschauer, D. A. (1985, March). "Fiscal Policy and Aggregate Demand." *American Economic Review* 75(2), 117–27.

Benveniste, L. M. and J. A. Scheinkman (1979, May). "On the Differentiability of the Value Function in Dynamic Models of Economics." *Econometrica* 47(3), 727–732.

Hall, Robert E. (1978, October). "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence." *Journal of Political Economy* 86(5), 971–987.

Santos, Manuel S. (1991, September). "Smoothness of the Policy Function in Discrete Time Economic Models." *Econometrica* 59(5), 1365–1382.

Bailey, Martin, *National Income and the Price Level.* (McGraw-Hill, 1971).

Deaton, Angus, Understanding Consumption (NY: Oxford University Press, 1992).

Hall, Robert, *The Rational Consumer: Theory and Evidence* (Cambridge: MIT Press, 1990)

Kreps, David, *A Course in Microeconomic Theory* (Princeton: Princeton Univ Press, 1990), Appendix 2.