0.1 Non-linear IS-LM Comparative Statics

A model is more than a collection of structural equations: it is also a specification of the endogenous variables. For example, suppose (1) and (2) are the "structural" equations of a model. Equation (1) is an IS equation: it describes goods market equilibrium: aggregate production (Y) equals the aggregate demand (A) for that production.

$$Y = A[\overset{-}{i - \pi}, \overset{+}{Y}, \overset{+}{F}] \tag{1}$$

Here i is the nominal interest rate, π is the expected inflation rate, and F measures the "fiscal stance" (i.e., how expansionary fiscal policy is). The plus or minus sign above each argument indicates the sign of the response of the function to an increase in the argument.

Equation (2) is an LM equation: it describes money market equilibrium. The real money supply (m) in the economy must equal real money demand (L) in the economy.

$$m = L[\overset{-}{i}, \overset{+}{Y}] \tag{2}$$

Refer to equations (1) and (2) as the "structural" equations, without worrying too much about the nature of economic structure. These two structural equations can be used to generate various models. E.g., produce a textbook "Keynesian" model by taking Y and i to be the endogenous variables, or produce a textbook "Classical" model by taking m and i to be the endogenous variables.

First, consider the textbook Keynesian model. Assuming satisfaction of the assumptions of the implicit function theorem, there is an implied reduced form for the Keynesian model. The reduced form expresses the solution for each endogenous variables in terms of the exogenous variables. Represent this as

$$i = i[m, \pi, F] \tag{3}$$

$$Y = Y[m, \pi, F] \tag{4}$$

Note the conventional use of the letter i to represent both a variable (on the left) and a function (on the right). For blackboard algebra, this is common practice among economists and mathematicians, as it helps us keep track of which function is related to which variable. (Computer algebra usually requires us to adopt different symbols for variable names and function names.) In the absence an explicit functional form (e.g., linear) for the structural equations, there does not exist an explicit functional form for the reduced form. Nevertheless, qualitative information about the structural equations can imply qualitative statements about the reduced form.

The implicit function theorem relates the partial derivatives of the reduced forms i and Y to such qualitative structural information. It provides the conditions under which the partial derivatives of the reduced form are functions of the partial derivatives of the structural form. These conditions are sufficient to investigate the qualitative comparative statics of the sturctural model.

First consider the money market. Recall that equation (2) described money market equilibrium as

$$m = L[i, Y]$$

Suppose a money market that is in equilibrium experiences an exogenous change and returns to equilibrium. The equation (2) must hold both before and after any exogenous changes. That is, there is equilibrium in the money market before the change, and there is equilibrium in the money market after the change. It follows that any change in the real money supply ($\mathrm{d}m$) must equal any change in real money demand ($\mathrm{d}L$). Represent the change in m as $\mathrm{d}m$ and the change in L as $\mathrm{d}L$. Then

$$\mathrm{d}m = \mathrm{d}L \tag{5}$$

Money demand depends on the value of its arguments. This implies that there are two possible sources of change in money demand: a changes in i, and/or a change in Y. As usual, represent these as $\mathrm{d}i$ and $\mathrm{d}Y$. However, the change in real money demand depends not only on the size of the changes in these arguments, but also on how sensitive money demand is to each argument. Represent these sensitivities as L_i and L_Y. (This is just an alternative notation for the partial derivatives.) The total change in money demand is the sensitivity-weighted sum of the changes in its arguments.

$$\mathrm{d}L = L_i \mathrm{d}i + L_Y \mathrm{d}Y \tag{6}$$

Recall that (5) displays how the changes in m must relate to the changes in L if the system is to both begin and end in equilibrium. Therefore, in light of (5),

$$\mathrm{d}m = L_i \mathrm{d}i + L_Y \mathrm{d}Y \tag{7}$$

Call equation (7) the "total differential" of the LM equation (2). It makes a very simple statement: in order to start out in equilibrium and then end up in equilibrium, any changes in m must equal the change in L.

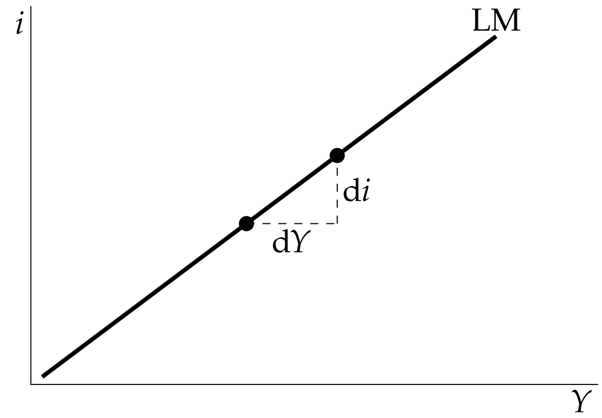

Figure 1: Slope of LM Curve

$$m = L[i, Y]$$

$$\mathrm{d}m = \mathrm{d}L$$

$$\mathrm{d}L = L_i \mathrm{d}i + L_Y \mathrm{d}Y$$

$$\mathrm{d}m = L_i \mathrm{d}i + L_Y \mathrm{d}Y$$

The total differential can be used to find the slope of the LM curve. Suppose we allow only i and Y to change (so that $dm = 0$). Then we must have

$$0 = L_i di + L_Y dY \tag{8}$$

$$\left.\frac{di}{dY}\right|_{LM} = -\frac{L_Y}{L_i} > 0 \tag{9}$$

This represents the way i and Y must change together to maintain equilibrium in the money market, ceteris paribus. That is, this determines the slope of the "Keynesian" LM curve. Under the standard assumptions that $L_Y > 0$ and $L_i < 0$, the "Keynesian" LM curve has a positive slope.

Next consider the goods market. Recall that the equation

$$Y = A[i - \pi, Y, F] \tag{1}$$

represents equilibrium in the goods market. This must hold both before and after any exogenous changes. That is, we require that we start out in goods market equilibrium, and we also require that we end up in goods market equilibrium. It follows that the changes in real income must equal the changes in real aggregate demand. Looking at the equation for the IS curve, we can see that this means that the change in real income (dY) must equal the change in real aggregate demand (dA).

$$dY = dA \tag{10}$$

The change in aggregate demand has three sources: changes in r, changes in Y, and changes in F. We represent these changes as dr, dY, and dF. Of course, the changes in aggregate demand depend not only on the size of the changes in these arguments, but also on how sensitive aggregate demand is to each of these arguments.

$$dA = A_r \overbrace{(di - d\pi)}^{dr} + A_Y dY + A_F dF \tag{11}$$

Putting these two pieces together, we have the total differential of the IS equation:

$$dY = A_r(di - d\pi) + A_Y dY + A_F dF \tag{12}$$

Note that $A[\cdot,\cdot,\cdot]$ has only three arguments. Do not be misled by the fact that we choose to write r as $i - \pi$. This does not change the number of arguments of the aggregate demand function. E.g., there is no derivative A_i. We may notice that if we allow only i

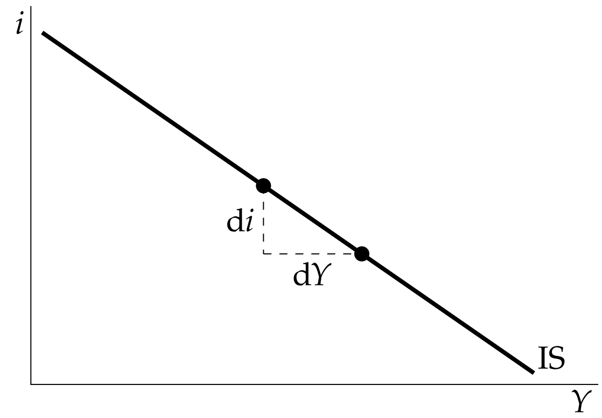

Figure 2: Slope of IS Curve

and Y to change, we must have

$$dY = A_r di + A_Y dY \tag{13}$$

$$\left.\frac{di}{dY}\right|_{IS} = \frac{1 - A_Y}{A_r} < 0 \tag{14}$$

This is the way i and Y must change together to maintain equilibrium in the goods market. That is, this determines the slope of the "Keynesian" IS curve. Under the standard assumptions that $0 < A_Y < 1$ and $A_r < 0$, the "Keynesian" IS curve has a negative slope.

So we have seen what is required to stay on the IS curve and what is required to stay on the LM curve. Putting these together we have

$$dY = A_r(di - d\pi) + A_Y dY + A_F dF \tag{15}$$

$$dm = L_i di + L_Y dY \tag{16}$$

When we insist that both of these equation hold together, we are insisting that we stay on both the IS and LM curves simultaneously. In this system there are two endogenous variables, dr and dY, which are being determined so as to achieve this simultaneous satisfaction of the IS and LM equations.

Well, you know how to solve two linear equations in two unknowns. First prepare to set up the system as a matrix equation by moving all terms involving the endogenous variables to the left. (Note that this is the first time we have paid attention to which variables are endogenous.)

$$-A_r di + dY - A_Y dY = -A_r d\pi + A_F dF \tag{17}$$

$$L_i di + L_Y dY = dm \tag{18}$$

Now rewrite this 2-equation system as a matrix equation in the form $\boldsymbol{J} \cdot \boldsymbol{x} = \boldsymbol{b}$, and then solve for the reduced form.

$$\begin{bmatrix} -A_r & (1 - A_Y) \\ L_i & L_Y \end{bmatrix} \begin{bmatrix} di \\ dY \end{bmatrix} = \begin{bmatrix} -A_r d\pi + A_F dF \\ dm \end{bmatrix} \tag{19}$$

Then solve for the endogenous variables by multiplying both sides by J^{-1}.

$$\begin{aligned} \begin{bmatrix} di \\ dY \end{bmatrix} &= \frac{1}{-A_r L_Y - (1 - A_Y) L_i} \begin{bmatrix} L_Y & -(1 - A_Y) \\ -L_i & -A_r \end{bmatrix} \begin{bmatrix} -A_r d\pi + A_F dF \\ dm \end{bmatrix} \\ &= \frac{1}{A_r L_Y + (1 - A_Y) L_i} \begin{bmatrix} -L_Y & (1 - A_Y) \\ L_i & A_r \end{bmatrix} \begin{bmatrix} -A_r d\pi + A_F dF \\ dm \end{bmatrix} \end{aligned} \tag{20}$$

Letting $\Delta = A_r L_Y + (1 - A_Y) L_i$, we can write this as

$$\begin{bmatrix} di \\ dY \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -L_Y & (1 - A_Y) \\ L_i & A_r \end{bmatrix} \begin{bmatrix} -A_r d\pi + A_F dF \\ dm \end{bmatrix} \tag{21}$$

Notice that $\Delta < 0$.

We know from the implicit function theorem that this is the same as solving for the partial derivatives of the reduced form. Invoking the standard assumptions on the structural form partial derivatives, listed above, we note that $\Delta = A_r L_Y + (1 - A_Y) L_i < 0$, we can write

$$\begin{aligned}\begin{bmatrix} \partial i / \partial m \\ \partial Y / \partial m \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} -L_Y & (1 - A_Y) \\ L_i & A_r \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} (1 - A_Y) \\ A_r \end{bmatrix} = \begin{bmatrix} - \\ + \end{bmatrix}\end{aligned} \tag{22}$$

$$\begin{aligned}\begin{bmatrix} \partial i / \partial \pi \\ \partial Y / \partial \pi \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} -L_Y & (1 - A_Y) \\ L_i & A_r \end{bmatrix} \begin{bmatrix} -A_r \\ 0 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} L_Y A_r \\ -L_i A_r \end{bmatrix} = \begin{bmatrix} + \\ + \end{bmatrix}\end{aligned} \tag{23}$$

$$\begin{aligned}\begin{bmatrix} \partial i / \partial F \\ \partial Y / \partial F \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} -L_Y & (1 - A_Y) \\ L_i & A_r \end{bmatrix} \begin{bmatrix} A_F \\ 0 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} -L_Y A_F \\ L_i A_F \end{bmatrix} = \begin{bmatrix} + \\ + \end{bmatrix}\end{aligned} \tag{24}$$

0.2 Classical Model

In the Classical case we follow the same procedures and the same type of reasoning, making only a single change: instead of Y we take m to be endogenous, so that m and i are the endogenous variables. Note that we start with the *same* system of structural equations:

$$Y = A[i - \pi, Y, F] \tag{25}$$

$$m = L[i, Y] \tag{26}$$

It follows that the total differential is unchanged:

$$\mathrm{d}Y = A_r(\mathrm{d}i - \mathrm{d}\pi) + A_Y\,\mathrm{d}Y + A_F\,\mathrm{d}F \tag{27}$$

$$\mathrm{d}m = L_i\,\mathrm{d}i + L_Y\,\mathrm{d}Y \tag{28}$$

Of course, all the partial derivatives from the structural form are unchanged: $A_r < 0$, $0 < A_Y < 1$, $A_F > 0$, $L_i < 0$, and $L_Y > 0$.

But of course we have a different set of endogenous variables, so we have a different implied reduced form:

$$\begin{aligned} m &= m[Y, \pi, F] \\ i &= i[Y, \pi, F] \end{aligned} \tag{29}$$

Since the set of endogenous variables has changes, we will rewrite our total differential to reflect the new set.

$$A_r\,\mathrm{d}i = A_r\,\mathrm{d}\pi + (1 - A_Y)\,\mathrm{d}Y - A_F\,\mathrm{d}F \tag{30}$$

$$-L_i\,\mathrm{d}i + \mathrm{d}m = L_Y\,\mathrm{d}Y \tag{31}$$

So when we write down the matrix equation, we use our new set of endogenous variables:

$$\begin{bmatrix} A_r & 0 \\ -L_i & 1 \end{bmatrix} \begin{bmatrix} di \\ dm \end{bmatrix} = \begin{bmatrix} A_r d\pi + (1 - A_Y)dY - A_F dF \\ L_Y dY \end{bmatrix} \tag{32}$$

Solving for the changes in the endogenous variables:

$$\begin{bmatrix} di \\ dm \end{bmatrix} = \frac{1}{A_r} \begin{bmatrix} 1 & 0 \\ L_i & A_r \end{bmatrix} \begin{bmatrix} A_r d\pi + (1 - A_Y)dY - A_F dF \\ L_Y dY \end{bmatrix} \tag{33}$$

So for example, consider the effects of a change in expected inflation, without any other exogenous changes. That is, let $\mathrm{d}\pi > 0$ but set $\mathrm{d}F = 0$ and $\mathrm{d}Y = 0$. This simplifies (34):

$$\begin{bmatrix} \mathrm{d}i \\ \mathrm{d}m \end{bmatrix} = \frac{1}{A_r} \begin{bmatrix} 1 & 0 \\ L_i & A_r \end{bmatrix} \begin{bmatrix} A_r \mathrm{d}\pi \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{d}\pi \\ L_i \mathrm{d}\pi \end{bmatrix} \tag{34}$$

Finally, divide through by $\mathrm{d}\pi$ to express the partial derivatives of the reduced form. (Note that we switch to curly deltas at this point.) The final matrix shows the qualitative response; we now need to rely on our knowledge of the signs of the partial derivative of L_i (as derived from economic theory).

$$\begin{bmatrix} \partial i / \partial \pi \\ \partial m / \partial \pi \end{bmatrix} = \begin{bmatrix} 1 \\ L_i \end{bmatrix} = \begin{bmatrix} 1 \\ - \end{bmatrix} \tag{35}$$