1 The Logistic Map

The term 'logistic map' refers to a set of quadratic functions defined on the unit interval.

$$f(x) = ax(1-x) \tag{1}$$

The parameter a is called the amplitude parameter. To ensure that $[0 .. 1] \mapsto [0 .. 1]$, we restrict the amplitude parameter to $a \in [0 .. 4]$. The plots in Figure 1 illustrate the effects on the function of changes in the amplitude parameter. Naturally, the larger the amplitude parameter, the larger is the maximum attained by the function.

1.1 Function Iteration

We can base a recurrence relation on (1):

$$x_{t+1} = ax_t(1-x_t) \tag{2}$$

Given an initial value x_0, we can produce a sequence of values through function iteration. We call this the trajactery of x_0 under the logistic map.

If the initial value is a fixed point, our trajectory will just be an endless repetition of that fixed point. Recall that fixed points satisfy $x = f(x)$. Since

$$x = ax(1-x) \implies ax(x-(1-1/a)) = 0 \tag{3}$$

we can clearly find a fixed point for this map at 0. If $a > 1$, we can find another at $1 - 1/a$. In Figure 1, find the second fixed point by looking for where the function plot crosses the 45-degree line.

What if our initial point is not a fixed point? Then the behavior of our sequence

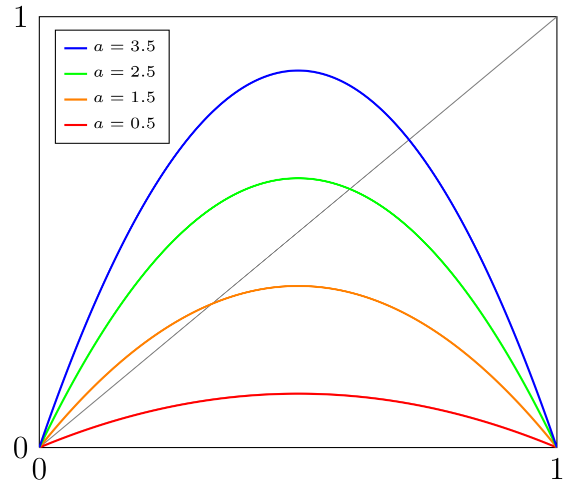

Figure 1: Logistic Map with Various Amplitudes

depends on the value of the amplitude parameter and possibly on the precise value we choose for x_0.

If $a \in (0 \mathrel{..} 1)$, then the iterative sequence converges to the single steady state at 0. But if $a \in (1 \mathrel{..} 2)$, then the iterative sequence converges to the second steady state at $1 - 1/a$. If $a \in (2 \mathrel{..} 3)$, then the iterative sequence still converges to the second steady state at $1 - 1/a$. Instead of converging monotonically, the sequence spends time on both sides of the steady state. This is because the steady-state value is now greater than $1/2$, which puts it on the other side of the function maximizer.

Convergence to the steady-state depends on the function slope at the steady state: it must be less than unity in absolute value. The slope is $f'(x) = a(1 - 2x)$. A fixed point x_{ss} is a stable equilibrium of our recurrence relation iff this slope is not too steep: we need $|f'(x_{ss})| < 1$. Note $f'(0) = a$ and $f'(1-1/a) = 2-a$. So 0 is a stable equilibrium if $a < 1$, and $1 - 1/a$ is a stable equlibrium if $1 < a < 3$. But for $a > 3$, neither is stable. So what happens then?

For $a \in (0 \mathrel{..} 1]$, we have one stable equilibrium at 0. Then we bifurcate: for $a \in (1, 3)$ there are two steady states, and the stable one is $1 - 1/a$.

The point $a = 3$ is another bifurcation point. The second steady-state passes into instability, but something else interesting happens. Let us illustrate by plotting 100 points of the trajectory on each side of $x = 3.0$. We will choose $a = 2.8$ and $a = 3.2$. In figure 3, $a = 2.8$ in the first panel, and $a = 3.2$ in the second panel.

After an initial "transient" fluctuation of about 15 points, both orbits "settle down" into a simple repetition. When $a = 2.8$, the trajectory approaches the single stable equilibrium. When $a = 3.2$, the trajectory becomes periodic, with period 2, oscillating between approximately 0.51 and 0.80.

We can be exact about this. Points of a two-period oscillation satisfy $x = f(f(x))$. That is, they are fixed points of $f^{\circ 2}$.

Note $f^{\circ 2}(x) = -a^3x^4 + 2a^3x^3 - a^3x^2 - a^2x^2 + a^2x$. Naturally the fixed points of f are also fixed points of $f^{\circ 2}$; that gives us two solutions for $x = f^{\circ 2}(x)$. Factor these out, and solve for two new fixed points

$$x = \frac{a(a+1) \pm a\sqrt{(a+1)(a-3)}}{2a^2} \tag{4}$$

However as we begin to raise a above 3, we find our old fixed points are now unstable, while the new points are stable.

To examine stability, we can look at the derivative

$$(f \circ f)' = -4a^3x^3 + 6a^3x^2 - 2a^3x - 2a^2x + a^2 \tag{5}$$

When $a = 3.2$, the slopes at the fixed points of $f \circ f$ are both 0.16.

Exercise 1

Of the 4 fixed points of $f \circ f$, which are stable at $a = 3.2$? Explain. Until what value of a do they remain stable? Then what happens as a increases?

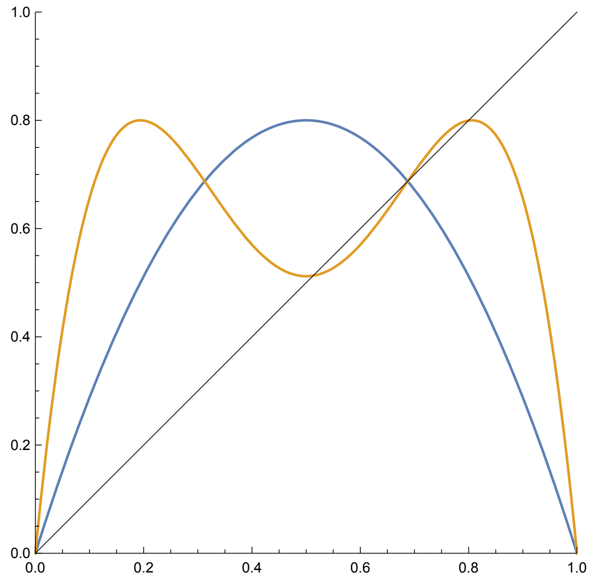

Figure 2: Logistic Function ($a = 3.2$): f and $f^{\circ 2}$

This period-2 orbit is stable or attracting in the following sense: independent of the initial condition in (0,1) every trajectory approaches a 2-period trajectory, fluctuating between these two points.

You should be suspicious of these pictures. If the arithmetic were exact, the orbit would approach the fixed points arbitrarily closely but never get there in finite time. (The initial point determines the actual deviations.) Since there is substantial "roundoff" error here due to the computer, we settle down completely, and pretty quickly. Furthermore, any display has a finite pixel size, so once the trajectory is sufficiently close to the equilibrium trajectory, it becomes visually indistinguishable from it.

So although the second fixed point of f becomes unstable, we discover a new stable structure appears: 2-period trajectory. We are going to futher explore what happens as a increases.

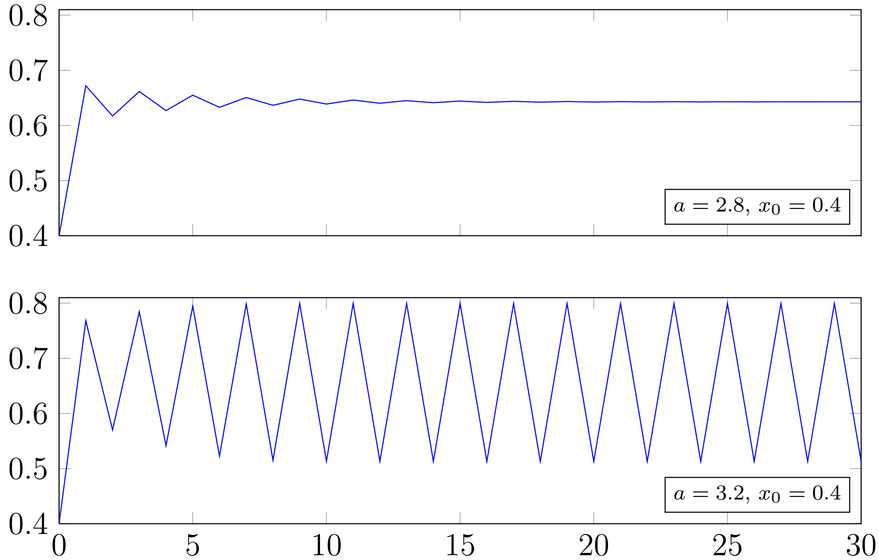

Figure 3: Emergence of a Period-2 Trajectory near $a = 3.0$

2 Cobweb Plots

A cobweb plot provides convenient two-dimensional visualization of one-dimensional dynamics. Given an intial point x_0, we know repeated function application will generate a sequence (x_t). For the sequence $(x_t)_{t\in[0\,..\,]}$, we will plot the following derived two-dimensional sequence: $((x_0, x_0), (x_0, x_1), (x_1, x_1), (x_1, x_2), (x_2, x_2), (x_2, x_3), \dots)$. In Figure 4, we set $a = 3.2$ and $x_0 = 0.25$ and plot the line segment between each of the implied. As is traditional, we also plot the segment from $(x_0, 0)$ to (x_0, x_0).

To give context to this sequence, we additionally plot the logistic function and the 45 degree line. So, start on the 45 degree line at $(0, x_0)$.[1] Draw a vertical connecting line to your function plot, at (x_0, x_1). Then draw a horizontal connecting line to the 45 degree line, at (x_1, x_1). Repeat this procedure as often as proves useful.

Now consider $a = 3.5$. As shown in figure 5, we settle down to a cycle: $(0.5009, 0.8750, 0.3828, 0.8269, \dots)$.

[1] Or alternatively at (x_0, x_0).

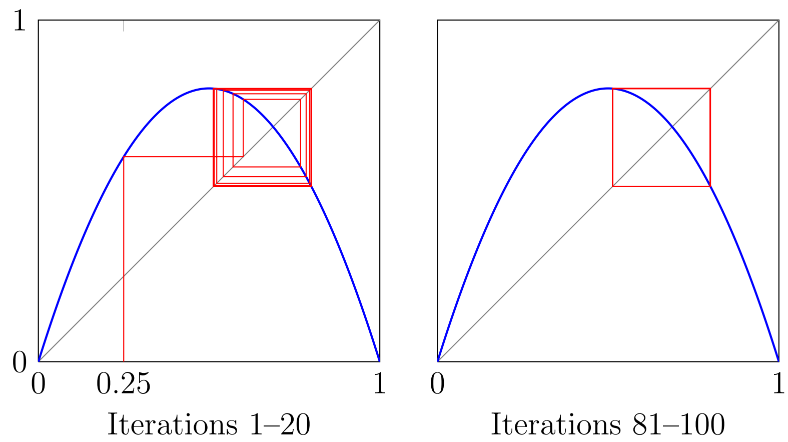

Figure 4: Cobweb Plot of Logistic Map ($a = 3.2$, $x_0 = 0.25$)

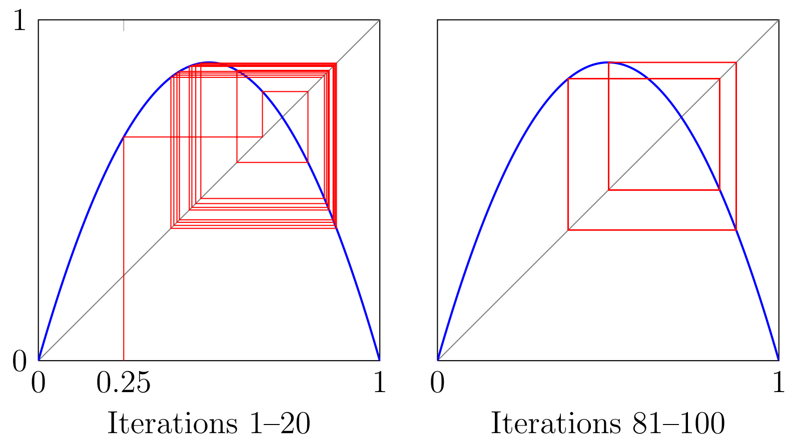

Figure 5: Cobweb Plot of Logistic Map ($a = 3.5$, $x_0 = 0.25$)

Exercise 2

Show that there is no stable orbit of period 3 for $a \in (3.0 .. 3.5)$.

As a increases, we get a "period-doubling cascade": the period 4 trajectory becomes unstable, but a stable period 8 trajectory appears; then the period 8 trajectory becomes unstable, but a stable period 16 trajectory appears. And so on. But something new also happens.

3 Bifurcation Diagram

We have found that the trajectory of the logistic map depends critically on the amplitude parameter. A bifurcation diagram offers a way to summarize this dependence. Figure 6 presents the bifurcation diagram from Wikipedia (by Jordan Pierce). The amplitude parameter varies along the horizontal axis. The stable fixed points of x are measure along the vertical axis. Each stable fixed point is marked with a point.

Here is how to produce such a diagram. For each $a \in (0 .. 4)$,

1. pick an arbitrary initial condition $x \in (0, 1)$. (The choice should not matter.)
2. iterate the logistic map enought times to eliminate transients (say, 1000 iterations)
3. discard all the transients (say, all but the last 50 iterations)
4. plot the unique remaining values of x against a.

For example, at $a = 3.5$ we know there is a stable period-4 trajectory, so we should have only 4 values of x to plot above $a = 3.5$.

Computational issues arise.

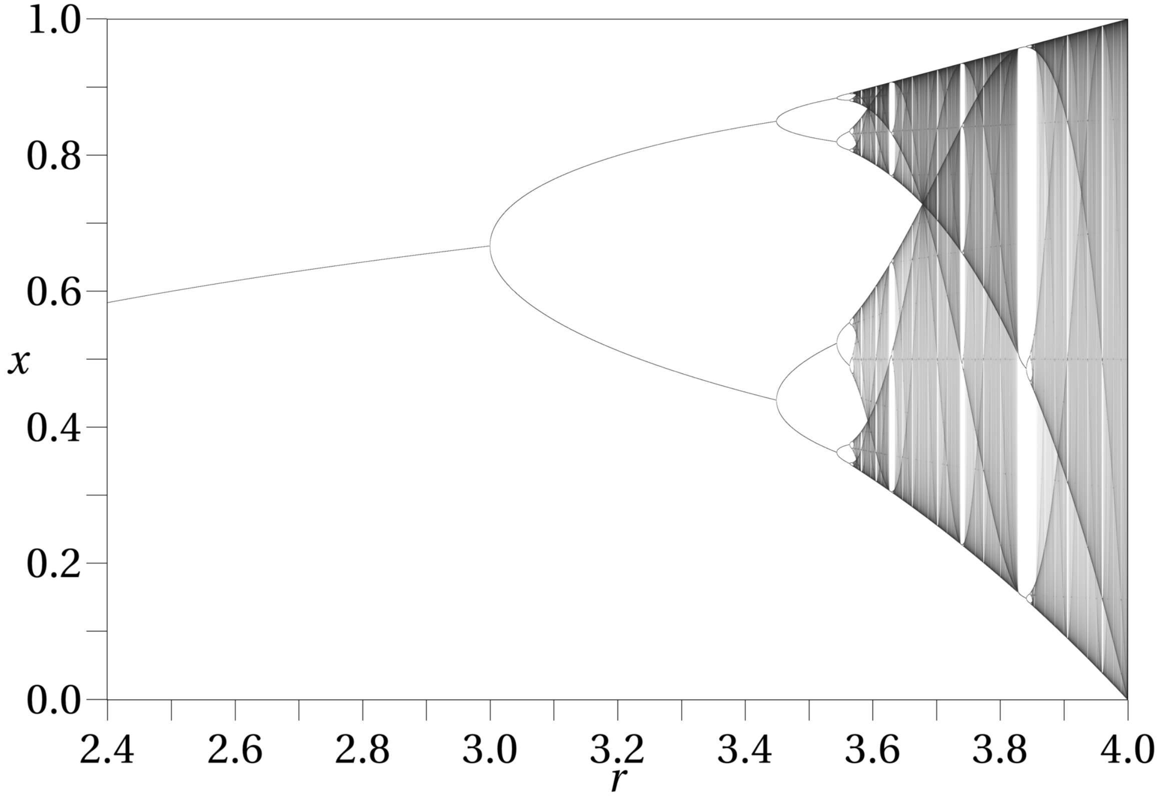

Figure 6: Bifrucation Diagram for Logistic Map