Lecture Notes 4

The Monetary Approach under Rational Expectations

International Economics: Finance

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We have seen that expectations of future inflation are an important determinant of the current exchange rate. This creates a very serious difficulty for research on exchange rate determination, since we know very little about expectations formation. In this course, we will consider several different ways that economists have struggled to grapple with this difficulty. The first approach we consider is the rational expectations hypothesis: the expectations of individuals are assumed to match the predictions of our model. It would more accurate to call these expectations model consistent, but the convention of calling them ‘rational’ is well established among economists. This handout offers a further exploration of the link between exchange rates and expectations, when expectations formation is “rational” in this restricted sense.

4.1 Expectations and Exchange Rates: The Monetary Approach

In handout 3 we explored the crucial role of expectations in the determination of exchange rates. In order to characterize the economic outcomes of anticipated monetary policy changes, we had to pin down the process of expectations formation. The approach we adopted was to assume expectations were a good match with actual outcomes. In a nutshell, that is the rational expectations hypothesis. The rational expectations hypothesis is that the expectations relevant to economic outcomes are appropriately proxied by the forecasts derived from the economist’s model.

To keep the algebra as simple as possible, we will work with the log-linear version of the monetary approach model. Recall that there are two basic components of the monetary approach to the determination of flexible exchange rates: purchasing power parity, and the classical model of price determination.

We begin with the log-linear representation of purchasing power parity.\(^1\)

\[
s_t = q_t + p_t - p_t^* \tag{4.1}
\]

As always, PPP says that the spot rate is proportional to the relative price level, where the factor of proportionality is the exogenous real exchange rate.

Next we consider the log-linear representation of the relative price level, as determined by the Classical model.\(^2\)

\[
p_t - p_t^* = h_t - h_t^* - [\phi(y_t - y_t^*) - \lambda(i_t - i_t^*)] \tag{4.2}
\]

As always, the Classical model tells us that the relative price level is determined by relative nominal money supplies and relative real money demands.

Together these two components yield our crude monetary approach model of exchange rate determination.

\[
s_t = q_t + h_t - h_t^* - \phi(y_t - y_t^*) + \lambda(i_t - i_t^*) \tag{4.3}
\]

\(^1\)For the moment, we are not imposing a constant real exchange rate, but we do continue to treat it as exogenous.

\(^2\)Here we work with the simplest version, which equates the foreign and domestic money demand parameters.
In handout 3 we looked at some of the empirical applications of this simple model. One potential problem for empirical tests of the crude monetary approach model is that it may mistake other influences on the exchange rate for a response to the interest rate. For example, the empirical version of the model allows for random shocks to money demand. Let $u_t$ be the money demand shock at time $t$, so that

$$s_t = q_t + h_t - h_t^* + \phi(y_t - y_t^*) + \lambda(i_t - i_t^*) - u_t$$

The money demand shocks ($u_t$) affect the current price level and thus the current exchange rate, but they also influence expected future inflation and thus the interest differential. For example, suppose $u_t$ is completely temporary (no serial correlation). Then a positive shock will increase money demand and lower the price level today, but next period the shock will be absent and the price level will rise again. So a positive money demand shock will have both a direct and indirect effect on the spot rate: by raising money demand it appreciates the exchange rate, but by raising expected future inflation (and the interest differential) it depreciates the exchange rate. Since the observable interest differential will be positively correlated with the unobserved money demand shocks, the estimate of $\lambda$ will be biased downward.\(^3\)

There are other difficulties as well. For example, there is the question of which interest rate should be used empirically, out of the array of possibilities. In this section we algebraically derive the predictions of the monetary approach to flexible rates under the rational expectations hypothesis. This allows us to overcome some of the problems that have concerned us, although it raises a few new questions as well.

We begin by recalling the covered interest parity condition from handout 2.

$$i_t - i_t^* = fd_t$$

In handout 2 we also decomposed the forward discount into two pieces: the expected rate of depreciation and the risk premium.

$$fd_t \equiv \Delta s^e_{t+1} + r_p_t$$

Recall that $\Delta s^e_{t+1}$ represents the expectation at time $t$ of the percentage rate of depreciation of the spot rate over the period $t$ to $t + 1$.

Equations (4.4) and (4.5) imply that the interest differential also bears a simple relationship to expected depreciation and the risk premium.

$$i_t - i_t^* = fd_t = \Delta s^e_{t+1} + r_p_t$$

We use the resulting expression to substitute for the interest differential in our crude monetary approach model. Substituting (4.6) into (4.3) yields

$$s_t = q_t + h_t - h_t^* + \phi(y_t - y_t^*) + \lambda(\Delta s^e_{t+1} + r_p_t) - u_t$$

\(^3\)This was noted by Driskill and Sheffrin (1981) and, in a similar context, Sargent (1977).
4.1.1 Fundamentals

Let us combine all the exogenous determinant of the exchange rate into a single variable, \( \tilde{m} \).

\[
\tilde{m}_t \overset{\text{def}}{=} q_t + h_t - h_t^* - \phi(y_t - y_t^*) + \lambda r p_t - u_t \tag{4.8}
\]

We will refer to \( \tilde{m} \) as the exchange rate fundamentals. This allows us to rewrite (4.7) in a slightly simpler form.

\[
s_t = \tilde{m}_t + \lambda \Delta s_{t+1} \tag{4.9}
\]

Equation (4.9) expresses the spot exchange rate in terms of the exchange rate fundamentals and expected depreciation. We see that an increase in expected depreciation causes the spot rate to depreciate: expectations are a crucial determinant of the spot rate. The spot rate is determined by the exchange rate fundamentals and expectations. Unless \( \lambda = 0 \), the fundamentals alone are not enough to determine the exchange rate.

Note that (4.9) implies that if no depreciation is expected, then exchange rate fundamentals directly determine the exchange rate (by determining the relative price level, of course). Changes in the exchange rate are very hard to predict, and when we have no reason to believe that depreciation or appreciation is more likely, our best guess may well be that it will remain unchanged. (That does not mean that we think the exchange rate will not change; it is just the guess that is best on average no matter how it actually does change.) In such cases we may say, roughly, that the exchange rate is believed to follow a random walk: each period it is just as likely to rise as to fall, and we have no reason to bet on a movement in one direction rather than another.

This situation presents obvious difficulties for empirical work: expectations are not observable. There are many proposals to deal with this, but we will only mention two. First, we might ask market participants what their expectations are. That is, we might rely on surveys. Although such suggestions seem to have recently gained some ground, traditionally economists have rejected this approach. Instead, there has been a tendency to finesse the observability problem by modeling expectations formation. At this point, most economists turn to the rational expectations hypothesis.

While (4.9) is helpful conceptually, it is not a complete solution of our model: expected depreciation can be decomposed into the current spot rate and the expected future spot rate.

\[
\Delta s_{t+1}^e = s_{t+1}^e - s_t \tag{4.10}
\]

We can therefore rewrite (4.9) as

\[
s_t = \tilde{m}_t + \lambda (s_{t+1}^e - s_t) \tag{4.11}
\]

Solving for the spot rate yields

\[
s_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} s_{t+1}^e \tag{4.12}
\]

Once again this solution drives home a key message: the current level of the spot rate depends on its expected future value.

Comment: Mark defines \( \gamma = 1/(1 + \lambda) \) and \( \psi = \lambda/(1 + \lambda) \).
4.2 The Monetary Approach under Rational Expectations

In handout 3, we derived predictions of the monetary approach by relying on a close relationship between expectations about the money supply and its actual evolution. Much of the empirical work on the monetary approach proceeds in this fashion, and the close relationship is often referred to as “rational” expectations. This section contains an algebraic presentation of the monetary approach to exchange rate determination under rational expectations.

The rational expectations hypothesis (REH) is that the expectations relevant to economic outcomes are appropriately proxied by the forecasts derived from the economist’s model. Let \( \mathcal{E}_t \) denote a mathematical expectation conditional on all information available at time \( t \) (including past values of the fundamentals and the structure of the model). Then for the monetary approach, we will represent the rational expectation hypothesis by

\[
se_{t+1} \overset{\text{def}}{=} \mathcal{E}_t s_{t+1} \quad \text{(4.13)}
\]

The REH renders the expected future exchange rate one of the variables explained by the model. This is an apparently simple way of endogenizing the expectations of the future, but it has very strong implications. We will show that it implies that the entire expected future of the economy is relevant to the current spot rate. Specifically, as noted by Bilson (1978, p.78), it implies that the current spot rate is a weighted sum of all expected future exchange rate fundamentals:

\[
s_t = \frac{1}{1 + \lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^i \mathcal{E}_t \tilde{m}_{t+i} \quad \text{(4.14)}
\]

Since the weights decline over time, just as in a discounted present value calculation, this is often referred to as the “present value” solution for the spot exchange rate.

Equation (4.14) is the solution for the spot exchange rate under rational expectations. Since it involves expectations of future fundamentals, it is not yet clear how we can use this solution in empirical work. Before addressing that question, you may want to work through the algebra that gives us (4.14). (Otherwise, skip ahead to section 4.3.)

4.2.1 The Rational Expectations Algebra

This section derives the spot rate solution (4.14).

Recall our solution (4.12) for the monetary approach model:

\[
s_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} s^e_{t+1} \quad \text{(4.12)}
\]

We will combine this with (4.13), our rational expectations hypothesis.

\[
s^e_{t+1} \equiv \mathcal{E}_t s_{t+1} \quad \text{(4.13)}
\]

The result is (4.15).

\[
s_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t s_{t+1} \quad \text{(4.15)}
\]
We are going to use (4.15) to express the spot rate in terms of fundamentals. Our first approach will use “recursive substitution.” We begin the observation that (4.15) applies at each point in time. Therefore it implies (4.16).

\[
    s_{t+1} = \frac{1}{1 + \lambda} \tilde{m}_{t+1} + \frac{\lambda}{1 + \lambda} \mathcal{E}_{t+1}s_{t+2}
\]

(4.16)

This is just the same relationship, one period forward in time. Taking expectations (at time \(t\)) of both sides of (4.15) yields equation (4.18).

\[
    \mathcal{E}_t s_{t+1} = \mathcal{E}_t \left\{ \frac{1}{1 + \lambda} \tilde{m}_{t+1} + \frac{\lambda}{1 + \lambda} \mathcal{E}_{t+1}s_{t+2} \right\}
\]

(4.17)

\[
    s_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t \tilde{m}_{t+1} + \left( \frac{\lambda}{1 + \lambda} \right)^2 \mathcal{E}_t \mathcal{E}_{t+1}s_{t+2}
\]

(4.18)

Of course, now we need to substitute for \(\mathcal{E}_{t+1}s_{t+2}\), and then \(\mathcal{E}_{t+2}s_{t+3}\) etc. Then we have after \(n\) substitutions we have

\[
    s_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t \tilde{m}_{t+1} + \cdots + \left( \frac{\lambda}{1 + \lambda} \right)^n \mathcal{E}_t \mathcal{E}_{t+1} \cdots \mathcal{E}_{t+n} \left\{ \frac{1}{1 + \lambda} \tilde{m}_{t+n} \right\}
\]

(4.19)

The Law of Iterated Expectations says that \(\mathcal{E}_t \{ \mathcal{E}_{t+i} s_{t+i+1} \} = \mathcal{E}_t s_{t+i+1}\). This just means that your current best guess of your future best guess about the future spot rate is just your current best guess about that future spot rate. In otherwords, your current guess uses all the information you have available, and therefore differs from your future best guess only due to new information you will receive in the future and cannot use now. This allows us to simplify (4.19).

\[
    s_t = \frac{1}{1 + \lambda} \sum_{i=0}^{n} \left( \frac{\lambda}{1 + \lambda} \right)^i \mathcal{E}_t \tilde{m}_{t+i} + \left( \frac{\lambda}{1 + \lambda} \right)^{n+1} \mathcal{E}_t s_{t+n+1}
\]

(4.20)

Noting that \(\lambda/(1 + \lambda) < 1\), we often assume \(\lim_{n \to \infty} [\lambda/(1 + \lambda)^{n+1}] \mathcal{E}_t s_{t+n+1} = 0\). Then we can write our solution as

\[
    s_t = \frac{1}{1 + \lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^i \mathcal{E}_t \tilde{m}_{t+i}
\]

(4.21)

\[\text{The situation is actually a bit more complicated than this suggests: see the discussion of “bubbles” in section ??}.\] For now, we assume the weighted sum of expected future fundamentals will converge.
4.2. RATIONAL EXPECTATIONS

Bubbles

Noting that $\lambda/(1+\lambda) < 1$, we assumed above that $\lim_{n\to\infty} [\lambda/(1+\lambda)^{n+1}] E_t s_{t+n+1} = 0$. However, we can imagine exchange rate “bubbles” that would generate a non-zero limit for this term. In this section we offer a simple illustration of this possibility.

Let $s^f_t$ be the present value solution. That is

$$s^f_t \overset{\text{def}}{=} \frac{1}{1+\lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^i E_t \tilde{m}_{t+i} \tag{4.22}$$

Next define a bubble $b_t$ as the following explosive first-order autoregressive process:

$$b_t = \frac{1+\lambda}{\lambda} b_{t-1} + \eta_t \tag{4.23}$$

Here $\eta_t$ is white noise (i.e., mean zero and constant variance). So

$$E_t b_{t+1} = \frac{1+\lambda}{\lambda} b_t \tag{4.24}$$

We propose that $s^f_t + b_t$, which we will call the bubble solution, is a solution to our rational expectations model. We check this by seeing if it satisfies (4.15). That is, does our proposed solution satisfy (4.15)?

$$s^f_t + b_t \overset{?}{=} \frac{1}{1+\lambda} \tilde{m}_t + \frac{\lambda}{1+\lambda} E_t \left( s^f_{t+1} + b_{t+1} \right) \tag{4.25}$$

$$b_t \overset{?}{=} \frac{\lambda}{1+\lambda} E_t b_{t+1} \tag{4.26}$$

$$b_t = b_t \tag{4.27}$$

The last equality follows from (4.24). So we see that the bubble solution is also a solution to our model.

Comment: In his related discussion, Mark uses $\hat{s}$ instead of $s$ and $\bar{s}$ instead of $s^f$.

4.2.2 The Role of News

Consider the implication of (4.21) for the difference between the realized spot rate and the anticipated spot rate. Taking expectations at time $t-1$ yields

$$E_{t-1}s_t = \frac{1}{1+\lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^i E_{t-1} \tilde{m}_{t+i} \tag{4.28}$$

Subtracting (4.28) from (4.21) yields (4.29).

$$s_t - E_{t-1}s_t = \frac{1}{1+\lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^i (E_t \tilde{m}_{t+i} - E_{t-1} \tilde{m}_{t+i}) \tag{4.29}$$

That is, the difference between the spot rate differs from its anticipated value to the extent that expectations about the fundamentals have been revised.
4.2.3 Another Representation of the Algebra

Use of the “forward shift operator” offers a simplified representation of the algebra. To reduce notational clutter, let \( \mu = \lambda/(1 + \lambda) \) and rewrite (4.15), our characterization of the spot exchange rate in terms of fundamentals and expectations, as (4.30).

\[
\begin{align*}
st & = \mathcal{E}_t \{(1 - \mu)\tilde{m}_t + \mu s_{t+1}\} \\
& = (1 - \mu)\tilde{m}_t + \mu \mathcal{E}_t s_{t+1}
\end{align*}
\] (4.30)

Since the expectation at time \( t \) of the current spot rate is just the current spot rate, and similarly for the current money supply, we can rewrite this as

\[
\mathcal{E}_t s_t = (1 - \mu)\mathcal{E}_t \tilde{m}_t + \mu \mathcal{E}_t s_{t+1}
\] (4.31)

Now consider the forward shift operator, \( F \), defined as \( x_{t+n} = F^n x_t \), and use it to rewrite (4.31) as

\[
(1 - \mu F) \mathcal{E}_t s_t = (1 - \mu)\mathcal{E}_t \tilde{m}_t
\] (4.32)

Now if we could just multiply both sides by the inverse of \( (1 - \mu F) \), we would have a solution for the spot rate.\(^5\)

\[
\mathcal{E}_t s_t = (1 - \mu F)^{-1} (1 - \mu)\mathcal{E}_t \tilde{m}_t
\] (4.33)

It turns out that we can do this. Define the inverse to be\(^6\)

\[
(1 - \mu F)^{-1} = 1 + \mu F + \mu^2 F^2 + \cdots \sum_{i=0}^\infty \mu^i F^i
\] (4.34)

So we can write our reduced form for the spot exchange rate as

\[
\mathcal{E}_t s_t = (1 - \mu) \sum_{i=0}^\infty \mu^i \mathcal{E}_t \tilde{m}_{t+i}
\] (4.35)

Of course \( \mathcal{E}_t s_t = s_t \), so we have found our solution for the spot rate.

\(^5\)More accurately, we would have

\[
\mathcal{E}_t s_t = (1 - \mu F)^{-1} (1 - \mu)\mathcal{E}_t \tilde{m}_t + \eta \mu^{-t}
\]

The term \( \eta \mu^{-t} \) is permitted in the solution, since

\[
(1 - \mu F) \eta \mu^{-t} = 0
\]

However we will ignore such “bubbles” in the solution.

\(^6\)Note that \( (1 - \mu F) (1 + \mu F + \mu^2 F^2 + \cdots) = 1. \)
4.3 An Observable Solution

We have seen that the exchange rate solution is a weighted sum of expected future fundamentals. Let us turn to the question of how to use this solution in empirical work. How can we handle the expected future fundamentals in the exchange rate solution? Until we can relate these expected future values to something we can observe and measure, our spot rate solution under rational expectations cannot be turned into a useful empirical model. For example, we cannot offer any simple relationship between the spot rate and the money supply (as we attempted to do in handout 3) until we know how the expected future behavior of the money supply is related to its current and past behavior. Many economists approach this problem by characterizing the way the fundamentals evolve over time. Such a characterization is called a data generating process (DGP) for the fundamentals. This information can then be used in forming expectations about the future.

A common assumption in early empirical tests of the monetary approach is that the exchange rate fundamentals follow a random walk:

$$\tilde{m}_t = \tilde{m}_{t-1} + u_t$$  \hspace{1cm} (4.36)

Here $u_t$ is the unanticipated change in the fundamentals, which averages zero (and is not serially correlated). This just means that the fundamentals are just as likely to rise as to fall each period. In this case our “best guess” of the future fundamentals is their current value. That is

$$\mathcal{E}_t \tilde{m}_{t+1} = \tilde{m}_t$$ \hspace{1cm} (4.37)

If the fundamentals are just as likely to rise as to fall, so is the spot exchange rate. That is, our best guess of the future spot rate is the current spot rate. Recalling (4.9), we can conclude that

$$s_t = \tilde{m}_t$$ \hspace{1cm} (4.38)

(This can also be seen algebraically by substituting (4.37) into (4.14).) Recalling the definition of the exchange rate fundamentals, we can then gather data and test (4.38) empirically.

$$s_t = q + h - h^* - \phi(y - y^*) + \lambda r$$ \hspace{1cm} (4.39)

Note how similar this is to the crude monetary approach: only the interest rate effect has disappeared.

Obviously the fundamentals cannot always be characterized as following a random walk. For example, if one country consistently has high inflation and another country consistently has low inflation, the random walk characterization of the exchange rate fundamentals for these two countries will be a poor one. As a result, for many exchange rates we need to characterize the exchange rate fundamentals by a more complicated data generating process. For example, we may assume that the fundamentals follow a simple autoregressive process.$^7$

$^7$A more typical way of closing a rational expectations model is much more general: assume that the $n$ exogenous variables follow a $k^{th}$-order vector autoregressive process (VAR) after differencing. This is a bit more complicated, so details will be relegated to appendix 1. Briefly, let $X_t$ be the $n$-vector of exogenous variables at time $t$. For example, we might treat relative money supplies and relative incomes as the only components of our fundamentals. Let $X^\top_t = (h - h^*, y - y^*)$, so we can rewrite our fundamentals as
Even this considerably complicates the problem of representing and summing up the expected future fundamentals. One useful alternative method for dealing with more general DGPs is the method of undetermined coefficients, which is briefly treated below.

### 4.4 The Data Generating Process

Before starting on the algebra, let us get our bearings. Remember, we have a model of the spot exchange rate under rational expectations

$$ s = \frac{1}{1 + \lambda} (\tilde{m} + \lambda \varepsilon_t s) \quad (4.12) $$

that provides the basic structure of spot rate determination. We found that we could solve this model for the spot rate as

$$ s_t = \frac{1}{1 + \lambda} \sum_{i=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^i \varepsilon_t \tilde{m}_{t+i} \quad (4.40) $$

where $\tilde{m}$ represents the exchange rate fundamentals (e.g., relative money supplies and the determinants of relative real money demand). Now the problem with this kind of formulation is that it still involves unobserved expectations. While it is informative at the theoretical level, it is not very useful empirically. We would like to have an observable solution for the exchange rate, a solution in terms of variables we can observe and measure. To move from a solution like (4.40) to an observable solution, we represent the fundamentals by a data generating process (DGP).

For example, we initially treated $\tilde{m}$ as following a random walk, which yielded a simple observable solution for the exchange rate. But while this may be a good approximation for some pairs of countries, for others it will be terrible. (For example, one country might have persistent high inflation when the other does not.) So consider a more general DGP: the following simple “autoregressive” representation of the fundamentals.

$$ \tilde{m}_t = \mu_0 + \mu_1 \tilde{m}_{t-1} + \epsilon_t \quad (4.41) $$

We will show that given the DGP (4.41), the exchange rate solution is

$$ s_t = \frac{\lambda \mu_0}{1 + \lambda - \lambda \mu_1} + \frac{1}{1 + \lambda - \lambda \mu_1} \tilde{m}_t \quad (4.42) $$

$\tilde{m}_t = a^\top X_t$ where $a^\top = (1, -\phi)$. The VAR process simply relates $X_t$ to its past values.

$$ \Delta X_t = \sum_{j=1}^{k} B_j \Delta X_{t-j} + v_t \quad (61) $$

The $B_j$'s are the matrices of coefficients on the lagged exogenous variables and $v_t$ is a vector of errors. If we create a vector $Z_t$ containing the current and lagged exogenous variables, then as shown in appendix .1, our solution for the exchange rate is a linear function of the current and lagged exogenous variables.

$$ s_t = a^\top X_t + a^\top GC \Delta Z_t \quad (70) $$

where $GC$ is a matrix defined in appendix .1. This can be estimated simultaneously with (61).
4.4. THE DATA GENERATING PROCESS

4.4.1 Anticipating Future Fundamentals

Now if we know the DGP (4.41) governs the evolution of the fundamentals over time, then in the future we will have

\[ \tilde{m}_{t+1} = \mu_0 + \mu_1 \tilde{m}_t + u_{t+1} \]
\[ \tilde{m}_{t+2} = \mu_0 + \mu_1 \tilde{m}_{t+1} + u_{t+2} \]
\[ \tilde{m}_{t+3} = \mu_0 + \mu_1 \tilde{m}_{t+2} + u_{t+3} \]

etc.

Forming our expectations in light of this knowledge tells us

\[ \mathcal{E}_t \tilde{m}_{t+i} = \mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_t + \mathcal{E}_t u_{t+i} \]
\[ = \mu_0 + \mu_1 \tilde{m}_t \]
\[ \mathcal{E}_t \tilde{m}_{t+2} = \mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_{t+1} + \mathcal{E}_t u_{t+2} \]
\[ = \mu_0 + \mu_1 (\mu_0 + \mu_1 \tilde{m}_t) \]
\[ = \mu_0 + \mu_1 \mu_0 + \mu_1^2 \tilde{m}_t \]
\[ \mathcal{E}_t \tilde{m}_{t+3} = \mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_{t+2} + \mathcal{E}_t u_{t+3} \]
\[ = \mu_0 + \mu_1 (\mu_0 + \mu_1 \mu_0 + \mu_1^2 \tilde{m}_t) \]
\[ = \mu_0 + \mu_1 \mu_0 + \mu_1^2 \mu_0 + \mu_1^3 \tilde{m}_t \]
\[
\vdots
\]

We quickly see a pattern emerging, which can be summarized by (4.43).

\[ \mathcal{E}_t \tilde{m}_{t+i} = \mu_0 \sum_{j=0}^{i-1} \mu_1^j + \mu_1^i \tilde{m}_t \quad i \geq 1 \quad (4.43) \]

This gives us an important insight. All of our expectations of future fundamentals can be stated in terms of the current fundamentals.

4.4.2 Finding An Observable Reduced Form: The Algebra

In principle we could substitute our solutions (4.43) for expected future fundamentals into equation (4.40) and actually perform the summation. This can be very messy, however, so often it is convenient to rely instead on the method of undetermined coefficients. We will do it both ways.
Direct Summation

\[ s_t = (1 - \mu) \sum_{i=0}^{\infty} \mu^i \mathcal{E}_i \tilde{m}_{t+i} \]

\[ = (1 - \mu) \sum_{i=1}^{\infty} \mu^i \mathcal{E}_i \tilde{m}_{t+i} + (1 - \mu) \tilde{m}_t \]

\[ = (1 - \mu) \sum_{i=1}^{\infty} \mu^i \left( \mu_0 \sum_{j=0}^{i-1} \mu_j^1 + \mu_1^i \tilde{m}_t \right) + (1 - \mu) \tilde{m}_t \]

\[ = (1 - \mu) \mu_0 \sum_{i=1}^{\infty} \mu^i \sum_{j=0}^{i-1} \mu_j^1 + (1 - \mu) \sum_{i=0}^{\infty} \mu^i \mu_1^i \tilde{m}_t \]

So we have

\[ s_t = \phi_0 + \phi_1 \tilde{m}_t \] \hspace{1cm} (4.45)

where \( \phi_0 \) and \( \phi_1 \) are the constant coefficients that are the infinite sums of structural form parameters in (4.44). It turns out that we can give much simpler representations of these coefficients.

Noting that

\[ \sum_{i=0}^{\infty} \mu^i \mu_1^i = \frac{1}{1 - \mu \mu_1} \] \hspace{1cm} (4.46)

we have

\[ \phi_1 = \frac{1 - \mu}{1 - \mu \mu_1} \] \hspace{1cm} (4.47)

Finding \( \phi_0 \) is a little more work. Let us begin with the inner summation.

\[ \sum_{j=0}^{i-1} \mu_j^1 = \frac{1}{1 - \mu_1} (1 - \mu_1^i) \] \hspace{1cm} (4.48)

Then we can write

\[ \phi_0 = \frac{1 - \mu}{1 - \mu \mu_1} \mu_0 \sum_{i=1}^{\infty} \mu^i (1 - \mu_1^i) \] \hspace{1cm} (4.49)

Now note that

\[ \sum_{i=1}^{\infty} \mu^i (1 - \mu_1^i) = \sum_{i=1}^{\infty} \mu^i - \sum_{i=1}^{\infty} \mu^i \mu_1^i \]

\[ = \frac{\mu}{1 - \mu} - \frac{\mu \mu_1}{1 - \mu \mu_1} \]

\[ = \frac{\mu(1 - \mu_1)}{(1 - \mu)(1 - \mu \mu_1)} \]

\[ = \frac{\mu_0}{1 - \mu} \] \hspace{1cm} (4.50)

So we get

\[ \phi_0 = \frac{1 - \mu}{1 - \mu_1} \mu_0 \frac{\mu(1 - \mu_1)}{(1 - \mu)(1 - \mu \mu_1)} \]

\[ = \frac{\mu \mu_0}{1 - \mu \mu_1} \] \hspace{1cm} (4.51)
Finally, since we have solved for $\phi_0$ and $\phi_1$, we can write

$$s_t = \frac{\mu_0}{1 - \mu_1} + \frac{1 - \mu}{1 - \mu_1} \tilde{m}_t$$  \hspace{1cm} (4.52)

The Method of Undetermined Coefficients

Our second approach to finding an observable reduced form will be the method of undetermined coefficients. (Naturally, it will yield the same solution.) The method of undetermined coefficients requires that we guess the general form of our solution. However the guess is not arbitrary. Instead it relies on two pieces of information. First, we saw in equation (4.40) that we can express the current spot rate in terms of expected future fundamentals. Second, we found that all expected future fundamentals can be expressed in terms of current fundamentals. These two pieces of information suggest that the spot rate can be expressed in terms of the current fundamentals. This is captured by the following expression.

$$s_t = \phi_0 + \phi_1 \tilde{m}_t$$  \hspace{1cm} (4.53)

Unfortunately, this expression involves two undetermined coefficients, $\phi_0$ and $\phi_1$. We would like to know how these are related to the underlying structural parameters. We can get information about this by turning once again to equation (4.12). First note that our guessed solution implies

$$s_{t+1} = \phi_0 + \phi_1 \tilde{m}_{t+1}$$  \hspace{1cm} (4.54)

So

$$\mathcal{E}_t s_{t+1} = \phi_0 + \phi_1 \mathcal{E}_t \tilde{m}_{t+1}$$  
$$= \phi_0 + \phi_1 (\mu_0 + \mu_1 \tilde{m}_t)$$  \hspace{1cm} (4.55)
So we can substitute our expressions for $s$ and $s^e$ in the undetermined coefficients into equation (4.12) to get

$$s_t = \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} \mathcal{E}_t s_{t+1}$$

$$\phi_0 + \phi_1 \tilde{m}_t \equiv \frac{1}{1 + \lambda} \tilde{m}_t + \frac{\lambda}{1 + \lambda} [\phi_0 + \phi_1 (\mu_0 + \mu_1 \tilde{m}_t)]$$

$$= \frac{\lambda}{1 + \lambda} (\phi_0 + \phi_1 \mu_0) + \frac{1 + \lambda \phi_1 \mu_1}{1 + \lambda} \tilde{m}_t$$

(4.57)

Now here is the tricky part. Equation (4.57) must hold for any level of the fundamentals $\tilde{m}_t$. That is what allows us to determine the $\phi$'s: the slope and intercept must be the same for the function of $\tilde{m}_t$ that we find on each side of the equality (4.57).

$$\phi_0 \equiv \frac{\lambda}{1 + \lambda} (\phi_0 + \phi_1 \mu_0)$$

$$\phi_1 \equiv \frac{1 + \lambda \phi_1 \mu_1}{1 + \lambda}$$

(4.58)

These are two equations in the two unknowns $\phi_0$ and $\phi_1$. The second equation can be solved for

$$\phi_1 = \frac{1}{1 + \lambda - \lambda \mu_1}$$

(4.59)

so the first gives us

$$\phi_0 = \lambda \mu_0 \phi_1$$

$$= \frac{\lambda \mu_0}{1 + \lambda - \lambda \mu_1}$$

So we can now replace the undetermined coefficients in our guess (4.53) to get

$$s_t = \frac{\lambda \mu_0}{1 + \lambda - \lambda \mu_1} + \frac{1}{1 + \lambda - \lambda \mu_1} \tilde{m}_t$$

(4.60)

In empirical work, we can then estimate (4.60) and (4.41) simultaneously.

Equation (4.60) has an important message for exchange rate research. There is no simple relationship between exchange rates and money supplies. Note that the relationship between the spot rate and the money supply depends on all the parameters of the data generating process. For example, if $\mu_1 < 1$ so that money supply increases tend to be reversed over time, then the exchange rate will rise less than in proportion to a money supply increase.

In summary, the method of undetermined coefficients proceeds as follows:

1. Begin with a model and a data generating process (DGP) for the exogenous variables.

2. Based on the DGP, make an educated guess about the functional form of the solution, which will generally involve the exogenous variables, shocks to the system, and possibly lagged endogenous variables. That is, guess an “observable reduced form” for the system, where your guess involves “undetermined coefficients” that you want to express in terms of structural coefficients.
4.5. AN EMPIRICAL APPLICATION

3. Find the expectations implied by your proposed solution form.

4. Plug these expectations into the model.

5. Use the implied identities in the coefficients to solve for the undetermined coefficients in terms of the structural coefficients (here, the $\mu$s and $\lambda$).

4.5 An Empirical Application

The monetary approach to flexible exchange rates has been tested under the assumption of rational expectations. After some early results lent some support to the model, a great deal of testing took place. Three salient supportive studies are Hoffman and Schlagenhauf (1983), MacDonald (1983), and Woo (1985). Hoffman and Schlagenhauf model fundamentals individually as ARIMA(1,1,0) and get pretty good results (1974.1-1979.12) for the dollar vs. the franc, pound, and mark. Woo follows Bilson (1978) in incorporating lagged real balances in the money demand equation.

Hoffman and Schlagenhauf (1983) offered an early application of this rational expectations version of the monetary approach to exchange rate determination. Using monthly data, they applied the model to the exchange rate of the dollar for the Deutsche Mark, French Franc, and British Pound for 1974.06–1979.12. Typically, they do not estimate the relative money supply coefficient—predicted by the monetary approach to be unity—in their rational expectations model. However, they do test this restriction jointly with several others, and fail to reject the restrictions. Since under the rational expectations hypothesis the structural model and DGP combine to produce a number of non-linear parameter restrictions, as we saw in (70), their failure to reject these restrictions statistically was viewed as very interesting support for the rational expectations version of the monetary approach. Their estimates for the income elasticity of money demand and the interest rate semi-elasticity of money demand are reported in table 4.1.

<table>
<thead>
<tr>
<th>Sensitivity to:</th>
<th>Exchange Rate</th>
<th>USD/DEM</th>
<th>USD/FRF</th>
<th>USD/GBP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal interest rates</td>
<td>1.784</td>
<td>1.108</td>
<td>1.782</td>
<td></td>
</tr>
<tr>
<td>Real income</td>
<td>1.507</td>
<td>0.969</td>
<td>0.508</td>
<td></td>
</tr>
</tbody>
</table>

Source: Hoffman and Schlagenhauf (1983)

Table 4.1: Monetary Approach Estimates under Rational Expectations

As suggested by the monetary approach, these estimates are all positive; they are also of plausible magnitude. Finally, Hoffman and Schlagenhauf found all of them significant at the

\[ \Delta s_t = a' \Delta X_t + a' G C \Delta^2 Z_t \]

---

8They restricted the data generating process for the exogenous variables to be AR(1) for each variable individually (in differences). This imposes structure on our $C$ matrix (see the appendix). Also, they estimated a differenced form of the exchange rate equation:
5% level. All in all, this early study appears to provide impressive support for the rational expectations version of the monetary approach to flexible exchange rates.

4.6 Concluding Comments

A key lesson of this chapter is that there is no simple relationship between changes in the money supply and changes in the exchange rate. The effect of a change in the current money supply on the current exchange rate depends crucially on its effect on the expected future money supply. So beliefs about the monetary policy “reaction function” will be an important determinant of the contemporary link between exchange rates and money supplies. We will pursue this insight further in handout 11.

In the Classical model, nominal interest rates are determined by expected inflation. If individuals accurately link expected inflation to their anticipations of monetary policy, we can make more detailed predictions about the behavior of prices over time. This observation carries over immediately to the monetary approach to exchange rate determination, since in the monetary approach PPP ensures that any predictions we make about the behavior of prices are also predictions about the behavior of the exchange rate.
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Problems for Review

1. Suppose $\tilde{m}_t = \mu_0 + \mu_1 \tilde{m}_{t-1} + u_t$ where $u_t$ is an unexpected innovation in the data generating process. Then $\mathcal{E}_t \tilde{m}_{t+4}$ equals

(a) $\mu_0 + \mu_1 \mathcal{E}_t \tilde{m}_{t+3}$
(b) $\mu_0 (1 + \mu_1) + \mu_1^2 \mathcal{E}_t \tilde{m}_{t+2}$
(c) $\mu_0 (1 + \mu_1 + \mu_1^2) + \mu_1^3 \mathcal{E}_t \tilde{m}_{t+1}$
(d) $\mu_0 (1 + \mu_1 + \mu_1^2 + \mu_1^3) + \mu_1^4 \tilde{m}_t$
(e) All of the above.

2. Consider the monetary approach to flexible exchange rates under the assumption of “rational” expectations. The current spot exchange rate

(a) depends on all future fundamentals.
(b) depends on all expected future fundamentals.
(c) equals the current value of the fundamentals.
(d) b. and c.
(e) all of the above

3. Use the method of undetermined coefficients to solve for an observable reduced form of the monetary approach to flexible exchange rates under rational expectations. Use the following DGP: $m(t) = 0.8m(t-1) + u(t)$ where $u(t)$ is white noise (i.e., is zero on average). Show all the steps in your solution procedure.

4. Use the method of undetermined coefficients to solve for an observable reduced form of the monetary approach to flexible exchange rates under rational expectations. Use the following DGP: $m_t = 0.8m_{t-1} + u_t$ where $u_t = \rho u_{t-1} + v_t$ and $v_t$ is white noise. How does allowing $u_t$ to follow an autoregressive process change the solution?
Bibliography


.1 The Data Generating Process:
Detailed Analysis

A typical way of closing a rational expectations model is by assuming that the $n$ exogenous variables follow a $k$th-order vector autoregressive process (VAR) after differencing. The following treatment provides some general tools for implementing this closure.\(^9\) Let $X_t$ be the $n$-vector of exogenous variables at time $t$.

$$\Delta X_t = \sum_{j=1}^{k} B_j \Delta X_{t-j} + \nu_t$$

\(^9\)Much of the development below follows Driskill et al. (1992) fairly closely.
The $B_j$s are the $(n \times n)$ matrices $\{b_{ii,1}\}$, and $v_t$ is an $n$-vector of serially uncorrelated errors. Equation (61) can be rewritten as a first-order VAR as follows.

$$\Delta Z_t = A\Delta Z_{t-1} + \delta_t$$

(62)

where,

$$Z_t^\top = (X_{1,t}, \ldots, X_{1,t-k+1}, X_{2,t}, \ldots, X_{2,t-k+1}, \ldots, X_{n,t}, \ldots, X_{n,t-k+1})$$

$$\delta_t = (v_{1,t}, 0, \ldots, 0, v_{n,t}, 0, \ldots, 0, \ldots, v_{n,t}, 0, \ldots, 0)$$

and, $A$ is the $nk \times nk$ matrix whose $(i-1)k + 1^{th}$ row is the column vectorization of the matrix formed by vertically concatenating the $i^{th}$ row of all the $B_j$s, with the rest of the elements zero except for the $k - 1$ identity matrices beginning below each $b_{ii,1}$.

$$A = \begin{bmatrix}
  b_{11,1} & \cdots & b_{11,k} & b_{12,1} & \cdots & b_{12,k} & \cdots & b_{1n,1} & \cdots & b_{1n,k} \\
  1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  b_{n1,1} & \cdots & b_{n1,k} & b_{n2,1} & \cdots & b_{n2,k} & \cdots & b_{nn,1} & \cdots & b_{nn,k} \\
  0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
  0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 1 & 0 \\
  0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

We then calculate the $j$-step ahead linear least squares predictor $\mathcal{E}_t Z_{t+j}$ as follows. We can write,

$$\mathcal{E}_t Z_{t+j} = \mathcal{E}_t[Z_{t+j} - (Z_{t+j-1} + \ldots + Z_t) + (Z_{t+j-1} + \ldots + Z_t)]$$

$$= \mathcal{E}_t[\Delta Z_{t+j} + \Delta Z_{t+j-1} + \ldots + \Delta Z_{t+1} + Z_t]$$

$$= \sum_{i=0}^{j-1} \mathcal{E}_t \Delta Z_{t+j-i} + Z_t$$

(63)

Now from the data generating process (62), we know

$$\mathcal{E}_t \Delta Z_{t+j} = A \mathcal{E}_t \Delta Z_{t-1+j}$$

$$= A^2 \mathcal{E}_t \Delta Z_{t-2+j}$$

$$= \ldots$$

$$= A^j \Delta Z_t$$

(64)
Combining (63) with equation (64) we get;

\[ E_t Z_{t+j} = Z_t + \sum_{i=0}^{j-1} A^{j-i} \Delta Z_t \]

\[ = Z_t + \sum_{i=1}^{j} A^i \Delta Z_t \]

Now \(^{10}\)

\[ \sum_{i=1}^{j} A^i = A(I - A^j)(I - A)^{-1} \]

\[ \therefore E_t Z_{t+j} = Z_t + A(I - A^j)(I - A)^{-1} \Delta Z_t \quad (65) \]

Equation (65) is the \( j \)-step ahead linear least squares predictor of \( Z_t \).

Let \( G \) be a \((n \times nk)\) matrix of zeros, except for the elements \( g_{11}, g_{2,k+1}, g_{3,2k+1}, \text{ etc.} \) which are equal to one, then \( X_t = GZ_t \). Hence, using equation (65) we get the \( j \)-step ahead linear least squares predictor of \( X_t \) as follows.

\[ E_t (X_{t+j}) = GZ_t + GA(I_{nk} - A^j)(I_{nk} - A)^{-1} \Delta Z_t \quad (66) \]

One more preliminary. It will be useful to note that

\[ A \sum_{j=0}^{\infty} \mu^j (I - A^j)(I - A)^{-1} = A \left( \sum_{j=0}^{\infty} \mu^j I - \sum_{j=0}^{\infty} \mu^j A^j \right) (I - A)^{-1} \]

\[ = \left[ \frac{1}{1 - \mu} I - (I - \mu A)^{-1} \right] A(I - A)^{-1} \]

\[ = \frac{1}{1 - \mu} (I - \mu A)^{-1} [(I - \mu A) - (1 - \mu)I] A(I - A)^{-1} \]

\[ = \frac{1}{1 - \mu} (I - \mu A)^{-1} [\mu(I - A)] A(I - A)^{-1} \]

\[ = \frac{\mu}{1 - \mu} A(I - \mu A)^{-1} \]

Letting

\[ C = \mu A(I - \mu A)^{-1} \quad (68) \]

we can express this result as

\[ A \sum_{j=0}^{\infty} \mu^j (I - A^j)(I - A)^{-1} = \frac{1}{1 - \mu} C \quad (69) \]

\(^{10}\)This is the analogue to the scalar result. Recall

\[ \sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a} \]

The sum from 1 to \( j \) is \( a \) times the sum from 0 to \( j - 1 \).
The Observable Reduced Form

Recall that the semi-reduced form exchange rate equation was given by

$$s_t = (1 - \mu) \sum_{i=0}^{\infty} \mu^i \mathcal{E}_i \tilde{m}_{t+i}$$

where $$\tilde{m} \equiv h - h^* - \phi(y - y^*)$$ and $$\mu = \lambda / (1 + \lambda)$$. Let us now represent relative money supplies and relative incomes, our two exogenous variables, to follow a $$k$$-th order VAR, as discussed above. In the notation of the previous section, we have our exchange rate solution is

$$s_t = (1 - \mu) a' \sum_{i=0}^{\infty} \mu^i \mathcal{E}_i X_{t+i}$$

where

$$a = \begin{bmatrix} 1 & \phi \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad X_t = \begin{bmatrix} h - h^* \\ y - y^* \end{bmatrix}$$

From our results (66) and (69), we have

$$\sum_{i=0}^{\infty} \mu^i \mathcal{E}_i X_{t+i} = \sum_{i=0}^{\infty} \mu^i [GZ_t + GA(I_{nk} - A') (I_{nk} - A)^{-1} \Delta Z_t]$$

$$= \frac{1}{1 - \mu} G(Z_t + C \Delta Z_t)$$

We therefore have our solution for the exchange rate:

$$s_t = a' G(Z_t + C \Delta Z_t)$$

$$= a' X_t + a' G C \Delta Z_t$$

(70)

This can be estimated simultaneously with (61). Note that while the equation for the spot rate is linear in the exogenous variables, it is non-linear in the structural parameters (because of $$C$$). So if you wish to estimate the structural parameters, you will need to use a method that can account for this non-linearity.

Partial Adjustment of Money Demand

In this handout, we have continued to treat the current value of real money demand as a simple function of current income and interest rates. However, as noted in handout 3, and as Bilson (1978) and Woo (1985) emphasize, applied work on money demand generally uses this functional form only for “long-run” money demand. In handout 3, we showed that allowing for a partial adjustment characterization of money demand yields the spot-rate equation

$$s_t = q_t + (h_t - h_t^*) - \phi(y_t - y_t^*) + \lambda(s_{t-1}^e - s_{t-1} - rp) - \alpha(h_{t-1} - h_{t-1}^*) + \alpha s_{t-1} - \alpha q_{t-1}$$

When combined with absolute purchasing power parity and uncovered interest parity, we then have a new exchange rate equation

$$-\lambda s_{t-1}^e + (1 + \lambda) s_{t-1} - \alpha s_{t-1} = (h_t - h_t^*) - \phi(y_t - y_t^*) - \alpha(h_{t-1} - h_{t-1}^*)$$
We proceed with the solution under the assumption of rational expectations. Let $X_t^\top = (h_t - h_t^*, y_t - y_t^*)$, $a^\top = (-1, \phi)$, and $b^\top = (\alpha, 0)$. Then letting $\tilde{m}_t = a^\top X_t + b^\top X_{t-1}$ we can write

$$
\lambda \mathcal{E}_t s_{t+1} - (1 + \lambda) s_t + \alpha s_{t-1} = -(h_t - h_t^*) + \phi(y_t - y_t^*) + \alpha(h_t - h_t^*)
= a^\top X_t + b^\top X_{t-1}
= \tilde{m}_t
$$

Now transform this by taking expectations at time $t$.

$$
[\lambda F - (1 + \lambda) + \alpha F^{-1}] \mathcal{E}_t s_t = \mathcal{E}_t \tilde{m}_t \tag{71}
$$

With a slight abuse of notation, we will say the characteristic equation is

$$
\lambda F^2 - (1 + \lambda) F + \alpha = 0
$$

with solutions

$$
F_1, F_2 = \frac{1 + \lambda \pm \sqrt{(1 + \lambda)^2 - 4\lambda\alpha}}{2\lambda}
$$

Thus we can rewrite (71) as

$$(F - F_1)(F - F_2)\lambda F^{-1} \mathcal{E}_t s_t = \mathcal{E}_t \tilde{m}_t$$

which has the general solution

$$
\lambda F^{-1} \mathcal{E}_t s_t = (F - F_1)^{-1}(F - F_2)^{-1} \mathcal{E}_t \tilde{m}_t + c_1 F_1^t + c_2 F_2^t \tag{72}
$$

from the general solution, it is clear that the spot rate will tend to move explosively away from the fundamentals if either root is greater than unity in absolute value. Assuming that the parameters have the expected signs and magnitudes, $\lambda > 0$ and $\alpha \in (0, 1)$, there are two positive real roots.\textsuperscript{11} In addition, the smaller root is less than unity while the larger root is greater than unity. Probably the easiest way to see this is to note that $dF_1/d\alpha > 0$ and $dF_2/d\alpha < 0$, and then consider the values of $F_1$ and $F_2$ at the extreme values of $\alpha$.

### Characteristic Roots: Relative Magnitudes

<table>
<thead>
<tr>
<th>$0 &lt; \beta &lt; 1$</th>
<th>$\beta \geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{array}{cc} F_1 &amp; F_2 \end{array}$</td>
<td>$\begin{array}{cc} F_1 &amp; F_2 \end{array}$</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$\begin{array}{c} 0 \ 1 + 1/\beta \end{array}$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>$\begin{array}{c} 1 \ 1/\beta &gt; 1 \end{array}$</td>
</tr>
</tbody>
</table>

Thus we have a situation of saddle-path stability. Here is how we will deal with the instability deriving from $F_2$: we will set $c_2 = 0$. This is known as a “transversality condition”; it assures us that the exchange rate will approach its fundamentals in the long run. Putting it another way, it is ruling out explosive exchange rate bubbles. The transversality condition is serving another important role for us in our solution: without it, we do not have enough

\textsuperscript{11}With $\lambda, \alpha > 0$ the requirement that $(1 + \lambda)^2 - 4\lambda\alpha > 0$ is satisfied as long as $\lambda + 1/\lambda > 4\alpha - 2$. Given the nature of the partial adjustment mechanism (i.e., $\alpha < 1$), this is necessarily satisfied.
information to determine a unique exchange rate. That is because we are working with a second order difference equation, but we only have a single initial condition \((s_{t-1})\). We can highlight the role of this initial condition by multiplying (72) by \((F - F_1)\).

\[
(F - F_1)\lambda F^{-1}E_t s_t = (F - F_2)^{-1}E_t \tilde{m}_t
\]

Noting \(F_1F_2 = \alpha/\lambda\), we can write this as

\[
(1 - \frac{1}{F_1} F)\lambda F^{-1}E_t s_t = (1 - \frac{1}{F_2} F)^{-1}E_t \tilde{m}_t
\]

or, using summation notation,

\[
E_t s_{t-1} - \frac{1}{F_1} E_t s_t = \frac{\lambda}{\alpha} \sum_{i=0}^{\infty} \left( \frac{1}{F_2} \right)^i E_t \tilde{m}_{t+i}
\]

\[
E_t s_t = F_1E_t s_{t-1} - \frac{F_1}{\alpha} \sum_{i=0}^{\infty} \left( \frac{1}{F_2} \right)^i E_t \tilde{m}_{t+i}
\]

(73)

An Empirical Application

Woo (1985) estimates (73) jointly with (61) for the USD/DEM exchange rate over the period 1974.03–1981.10.\(^\text{12}\) He does not reject the restrictions imposed by the rational expectations hypothesis, and his money demand parameters have the expected signs and plausible magnitudes. Although his estimated money demand parameters were not always statistically significant, his work was taken to be very supportive of the monetary approach. The following table reports his parameter estimates with and without an autocorrelation correction.

<table>
<thead>
<tr>
<th>Woo (1985) Results</th>
<th>Uncorrected</th>
<th>Corrected</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>1.2964</td>
<td>1.2223</td>
</tr>
<tr>
<td>(\phi^G)</td>
<td>0.5924</td>
<td>0.4713</td>
</tr>
<tr>
<td>(\phi^{US})</td>
<td>0.3466</td>
<td>0.2867</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>0.7991</td>
<td>0.8570</td>
</tr>
</tbody>
</table>

\(^{12}\)In his VAR for the exogenous variables, Woo includes six lags on relative money supplies and domestic income but five lags on foreign income. He runs the VAR in levels rather than in differences, and he detrends the data to achieve stationarity.