These notes are *very* rough. Suggestions welcome.

Samuelson (1938, p.71) introduced revealed preference theory hoping to liberate the theory of consumer behavior "from any vestigial traces of the utility concept." The idea is that observed choices can reveal underlying preferences. The axioms of consumer theory therefore can be axioms of demand behavior—that is, of choices from budget sets—rather than of preferences or utility. Samuelson (1938) argued they should be, although by 1947 (Foundations) he allowed that the approaches could be complementary. As Pollak (1990, p.142) notes, however, the subsequent literature has emphasized the equivalence of demand axioms and preference axioms. This is closely linked to the antecedent integrability literature, which determined the conditions under which (inverse) demand functions can be integrated to obtain utility functions. These were symmetry ("mathematical integrability") and negative semi-definiteness ("economic integrability") of the Slutsky matrix.

Samuelson (1938) introduced his weak axiom of revealed preference (SWARP)—that a competitive consumer never reveals two bundles each to be preferred to the other—and showed that a consumer with well behaved preferences would always satisfy SWARP. (Here well behaved preferences are a monotonic, convex, continuous ordering.) However he left open the question as to whether SWARP exhausted the empirical content of the preference maximzation model. However, Houthakker (1950) introduced his strong axiom of revealed preference (HSARP) and showed that demand (i.e., choices from budget sets) obeys this axiom iff it can be generated by well-behaved preferences. Gale (1960) later clinched the case against sufficiency of SWARP by constructing a demand system that satisfied it but not HSARP.

HSARP states that competitive consumers never directly or indirectly reveal each of two bundles preferred to the other; that is, HSARP requires that the revealed preference relation be acyclical. As Pollak (1990, p.145) notes, that behavior satisfying preference maximization also satisfies HSARP is trivial. "Houthakker's accomplishment was showing the converse." That is, Houthakker answered the classic revealed preference riddle asking what restrictions on demand functions are equivalent to preference maximization on budget sets.

Despite Gale's result, the strong axiom is now often considered redundant. How did this come about? The key move was an extension of the domain of SWARP by Arrow (1959) to sets that are not budget sets. Specifically, Arrow interpreted revealed preference as the relationship the chosen element has to all other (rejected) alternatives in the choice set, whether or not the choice set is a budget set. He then argued that the same intuition that supported SWARP—that revealed inferior alternatives should never be chosen—should apply to choices from finite sets as well. Under this domain extension, the weak axiom implies the strong axiom. In fact, as Sen (1971) later showed, the weak axiom need only hold on all pairs and triads to imply the strong axiom.

Was Arrow correct that this domain extension is natural? Sen (1971) pursues this question. If the weak axiom is an *a priori* restriction on all rational behavior, then the extension does seem natural: why should it hold for budget sets but not for finite sets? Alternatively, perhaps it is more like a behavioral hypothesis than an axiom, in which case Sen allows that a restriction of scope is more defensible. But in this case, isn't the proper domain restriction to the finite number of observed choice situations? And in fact, Sen observed, the revealed preference axioms were not usually subjected to testing.

Interest in testing has increased, however, both for demand data and in experimental situations. This has generated interest in what Pollak has called the restricted domain version of revealed preference theory. The restricted domain version is directly interested in the information content of observed demand data. Therefore, it investigates the consistency of a finite number of choices from budget sets.

Key papers: Afriat (1967) (hard), Diewert (1973) (more accessible), Varian (1982) (the paper probably most responsible for popularizing Afriat's test.), Matzkin and Richter (1991).

The central result of Afriat was to show the equivalence of "cyclical consistency," essentially HSARP, and the existence of a solution to a linear programming problem. Thus Afriat, Diewert, and Varian offer a technique by means of which budget data can be examined directly for a general consistency with preference maximization. No specific functional form for the demand function (or, equivalently, the utility function) need be imposed on the data, as aspect of the technique that led Varian to refer to it as nonparametric.

Note that the alternative of estimating demand systems may not even allow the testing of consistency with preference maximization. For example, estimation of a Cobb-Douglas demand system that imposes the usual parameter restriction that budget shares are positive and sum to unity will always yield and estimated demand system consistent with preference maximization.

Three weaknesses of this approach noted by Pollak:

- treats choice as deterministic
- doesn't test anything if budget surfaces don't overlap (e.g., if expenditures grow much faster than prices change)
- if HSARP is in fact violated but SWARP is not, a very large number of observations may be required to establish the violation of HSARP. (Shafer, JET 16, 1977) That is, the test may have low power.

Nevertheless, the restricted domain version yields the conditions under which the preference maximization hypothesis can be refuted by a finite number of observations on competitive consumers.

There is a further difficulty for revealed preference theory to confront, as Pollak (1990) notes, and this concerns the dynamics of demand.

With naive habit formation, a consumer maximizes their one period utility function each period, failing to recognize the effect of current consumption on future preferences. But such behavior is rational only if preferences are intertemporally separable. A rational consumer will recognize the intertemporal dependence of consumption and preference, and will try to pick the best lifetime consumption plan [1] But then budget data has little hope of testing the preference maximization hypothesis, since for any individual's *lifetime* consumption choices really offer a single budget observation.

Ignoring this important problem, let's return to the question of testing the preference maximization hypothesis with finite observations. Matzkin and Richter (1991) have proved that HSARP tests the existence of a strictly concave, strictly monotone utility function in this setting. Combining this with our earlier results, we realize that for finite sets of observations on competitive consumers the preference maximization hypothesis is observationally equivalent to hypothesis that consumers maximize continuous, strictly concave, strictly monotone utility functions. (In fact, they show this is true even for pseudotransitive or semitransitve preverences.) Note that differentiability is *not* implied; however, as a practical matter we can have differentiability too (very roughly speaking).

[1] For interesting problems in this setting see Strotz (56), Pollak (1968), and Peleg and Yaari (1973).

1 Testing Rational Choice with Budget Data

Suppose we have a finite data set $\{(x^t, p^t)\}$ on consumer expenditures. We say that a utility function $U(\cdot)$ *rationalizes* the observed behavior if $U(x^t) > U(x) \quad \forall x \in B(p^t, P^t \cdot x^t)$.

Q: What observable restrictions on choices are implied by the existence of a rationalization?

A: None: all bundles may be indifferent. But given Walras' Law, there are restrictions.

Recall that Walras' Law implies i. $p^t \cdot x^t = y^t$

ii. $p^t \cdot x < p^t \cdot x^t \;\Rightarrow x \notin x(p^t, y^t)$

We will consider competitive consumers facing exogenous budget constraints of the form

$$B(p, y) = \{x \in \Re^K_+ \;\mid\; px \leq y\}$$

We observe a finite set of choices $x^1 \in C[B(p^1, y^1)], x^2 \in C[B(p^2, y^2)], x^3 \in C[B(p^3, y^3)], ..., x^n \in C[B(p^n, y^n)]$

Define weak revealed preference:

$$x^i \; WRP \; x^j \iff x^j \in B(p^i, y^i)$$

Define strict revealed preference:

$$x^i \; SRP \; x^j \iff p^i x^j < y^i$$

GARP: $\neg\exists x^{n1}, x^{n2} \ldots, x^{nk}$ s.t. $x^{n1} WRP x^{n2} WRP \ldots x^{nk} WRP x^{n1}$ AND one or more of the WRPs is a SRP.

The statement of Afriat's theorem follows Varian (1992, p.133).

Theorem 1 (Afriat's Theorem)
Given a finite set of consumer expenditure data $\{(x^t, p^t)\}$, the following conditions are equivalent:

1. *the data are consistent with maximization of a locally insatiable utility function*
2. *the data satisfy GARP*
3. *There exist positive numbers $\{(u^t, \lambda^t)\}$ satisfying the Afriat inequalities*
$$u^s \leq u^t + \lambda^t p^t \cdot (x^s - x^t) \quad \forall t, s$$

4. *There exists a locally nonsatiated, continuous, concave, monotonic utility function that rationalizes the data*

Proof:
i. That 4. implies 1. is trivial, and we have already seen that 1. implies 2.
ii. Varian (1982a) shows 2. implies 3.
iii. We show 3. implies 4. with a constructive proof.

Given positive $\{(u^t, \lambda^t)\}$ satisfying the Afriat inequalities, and $p^t, x^t \in \Re^K_+$, define

$$u(x) = \min_t \{u^t + \lambda^t p^t \cdot (x - x^t)\}$$

This function is continuous in x. For $p^t \gg 0$ this function is also strongly monotonic. To see that it is concave, note that it is just the lower envelope of a finite number of hyperplanes.

To show that the function rationalizes the data, first show that $U(x^t) = u^t$. Otherwise, we must have

$$u^s + \lambda^s p^s \cdot (x^s - x^t) < u^t$$

which violates the Afriat inequalities.

So whenver $p^s \cdot x^s \geq p^s \cdot x$ we know

$$U(x) = \min_t \{u^t + \lambda^t p^t \cdot (x - x^t)\} \leq u^s + \lambda^s p^s \cdot (x - x^s) \leq u^s = U(x^s)$$

So $U(x^s) \geq U(x)$ for all x such that $p^s \cdot x^s \geq p^s \cdot x$. That is, $U(\cdot)$ rationalizes the choices.

Comment: note the surprise in this result. A rationalization by a locally insatiable utility function implies the possibility of rationalizing with a LI, monotonic, continuous, *concave* utility function. In fact this has been strengthened by Matzkin and Richter to *strict* concavity (using single-valued demand)! Put another way, market data will never allow us to test for convexity and strict concavity of rational preferences.

1.1 Interpreting the Afriat Inequalities

Consider maximizatioin of a concave, differentiable utility function, assuming an interior maximum x^t. Then the F.O.C.s are

$$DU(x^t) = \lambda^t p^t$$

In addition, concavity implies

$$U(x) \leq U(x^t) + DU(x^t)(x - x^t)$$

Combining the FOCs and concavity yields

$$U(x) \leq U(x^t) + \lambda^t p^t (x - x^t)$$

So we see the Afriat numbers u^t and λ^t can be interpreted as utility levels and marginal utilities that are consistent with observed choices.

i. HW: LI$\Rightarrow$GARP

ii. HW: GARP $\Rightarrow$ representable and LI

Comment: since we observe only part of $C(A)$, the content of the theorem comes from the specification that preferences are locally insatiable. For example, if we were indifferent between all elements of X, then any observed outcomes would be consistent with preference max. GARP gives us necessary and sufficient conditions for a locally insatiable rationalization of observed demand behavior.

Definition: define demand to be *exhaustive* if $x \in C[B(p, y)]$ implies $x = py$.

Comment: note that GARP implies our choices are always exhaustive: $x^i = p^i y^i$. O/w, x^i *SRP* x^i.

Define direct revealed preference in the sense of Samuelson: xSx' iff x *WRP* x' and $x \neq x'$.

Definition: The *transitive closure* of a binary relation R is the smallest transitive binary relation containing R.

Let H be the transitive closure of S.

Definition: A binary relation B is *asymmetric* if $xBy \Rightarrow \neg yBx$.

Houthakker's strong axiom of revealed preference (SARP): H is asymmetric.

Matzkin and Richter (JET 1991) offer an extension of GARP. They show that if the choice function is single valued, then the following are equivalent statements about a finite set of budget data.

i. the data satify SARP

ii. the data can be rationalized by an strictly concave, strictly monotone function U.

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