Monotonic Saddle-Path Dynamics

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Abstract

This note provides necessary and sufficient conditions for the existence of a monotonic saddle path in second-order difference equations. The conditions are illustrated with three popular exchange-rate models.

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Introduction

Economically justifiable restrictions on structural models offer a powerful means of obtaining comparative static and dynamic results. Sometimes natural restrictions are sufficient to the needs of the analyst; other times supplementary restrictions must be invoked to prevent the model analysis from degenerating into pure taxonomy. For example, in most dynamic exchange-rate models, the existence of a monotonic saddle path is a desirable property. Unfortunately, the existence of such a saddle may be difficult or impossible to prove. In these circumstances it is not uncommon, and in many circumstances not unreasonable, to move the analysis forward by assuming the existence of a monotonic saddle path (e.g., Woo 1985; Papell 1988). This note provides necessary and sufficient conditions for the existence of a monotonic saddle path in second-order difference equations. Second-order difference equations are common in applied exchange-rate research, and the conditions are illustrated with three popular models.[1] The primary goal of this note is to provide and illustrate a simple test of whether desirable dynamic properties are implied by or must be imposed on a structural model.

Monotonic Saddle Paths

Standard discussions of difference equations concentrate on stability conditions, which are therefore well-known (e.g., Sargent 1979). However, economists are of-

[1] The field chosen for illustration is arbitrary, since the macroeconomic applications are broad. For example, the results in this note provide a quick route to Sargent's (1979, p. 198) illustrative saddle-path proof.

ten more interested in dynamic systems displaying "saddle-path stability". Consider the following linear, second-order difference equation with forcing function f_t and constant real coefficients b and c.

$$(F^2 + bF + c)x_t = f_t \tag{1}$$

Here F is the forward operator: $F^n x_t = x_{t+n}$. With a slight abuse of notation, we can write the characteristic equation as

$$F^2 - (F_1 + F_2)F + F_1 F_2 = 0 \tag{2}$$

where

$$F_1, F_2 = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right) \tag{3}$$

are the characteristic roots. The dynamics are of course characterized by a monotonic saddle path iff the characteristic roots are real and $0 < F_1 < 1 < |F_2|$.

Claim:

The difference equation (1) has a monotonic saddle path iff either $c > 0$ and $0 > 1 + b + c$, or $c < 0$, $1 + b + c > 0$, and $1 - b + c < 0$.

Proof:

CASE 1: $c > 0$

Necessity: We are given $0 < F_1 < 1 < F_2$. Therefore i. $c = F_1 F_2 > 0$ and ii. $1 + b + c = (1 - F_1)(1 - F_2) < 0$.

Sufficiency: We are given $c > 0$ and $0 > 1 + b + c$.

i. The roots are real since $\sqrt{b^2 - 4c} > \sqrt{(1+c)^2 - 4c} = \sqrt{(1-c)^2}$.

ii. The roots have the same sign since $F_1 F_2 = c > 0$ and therefore are both positive since $F_1 + F_2 = -b > 1 + c > 0$.

iii. Finally, $F_1 < 1 < F_2$ since $1 + b + c = (1 - F_1)(1 - F_2) < 0$.

CASE 2: $c < 0$

Necessity: We are given $F_2 < -1$ and $0 < F_1 < 1$. Therefore we know i. $c = F_1 F_2 < 0$, ii. $1 + b + c = (1 - F_1)(1 - F_2) > 0$, and iii. $1 - b + c = (1 + F_1)(1 + F_2) < 0$.

Sufficiency: We are given $c < 0$, $1 + b + c > 0$, and $1 - b + c < 0$.

i. We know $F_2 < 0 < F_1$ since $c = F_1 F_2 < 0$.

ii. We then know $F_1 < 1$ since $1 + b + c = (1 - F_1)(1 - F_2) > 0$.

iii. Finally $F_2 < -1$ since $1 - b + c = (1 + F_1)(1 + F_2) < 0$ (recall $F_1 > 0$).

Applications

Since the work of Meese and Rogoff (1983), applied work on structural exchange rate models has emphasized models that imply the presence of a lagged exchange rate in the estimated model. Three classic modifications of the basic "monetary" model achieve this: the partial adjustment approach of Woo (1985), the sticky-price approach of Dornbusch (1976) as implemented in Wickens (1985), and the portfolio effects approach as implemented by Driskill, Mark, and Sheffrin (1992). These models produce the following characteristic polynomials.

Characteristic Polynomials for Three Studies

Woo (1985)	$F^2 - \left(1 + \frac{1}{\beta}\right) F + \frac{\alpha}{\beta}$
Wickens (1985)	$F^2 - \left(1 + \mu + \frac{\theta}{\lambda}\right) F + \mu$
Driskill, Mark, and Sheffrin (1992)	$F^2 - \left(2 + \frac{\alpha}{\eta}\right) F + 1$

All parameters are as defined in the original studies, and theory assigns them positive values. As a result, it is immediate that the Wickens model and the Driskill, Mark, and Sheffrin model are saddle-path stable. In the Woo model, this result follows from the magnitude of α: it is a partial adjustment parameter, so $\alpha \in (0, 1)$. This means Woo (p.4) did not need to assume the existence of a saddle path in his model: the model structure implies it. There is also a corresponding implication in this result for the conduct of empirical work: if the estimated signs and magnitudes of the structural parameters conform to the models' predictions, then saddle-path behavior is empirically verified. For example, the results imply that Woo (p.7) need not have reported calculated values for F_1 and F_2: his estimated structural parameters conform to theory, rendering further study of the roots superfluous.

Conclusion

Economically justifiable restrictions on structural models offer a powerful means of obtaining comparative static and dynamic results. If natural restrictions are sufficient to the needs of the analyst, this can be verified using the necessary and sufficient conditions for the existence of a monotonic saddle path provided by this note. This application was illustrated with three popular exchange rate models, with implications for their theoretical and empirical presentations.

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